## 2.2. INVERSE Z TRANSFORM

The inverse z transform is the process of recovering the discrete time signal x (n) from its z transform X (z).

The inverse z transform can be determined by the following three methods.

1. Direct evaluation by contour integration (or residue method)
2. Partial fraction expansion method.
3. Power series expansion method

### 2.2.1. DIRECT EVALUATION BY CONTOUR INTEGRATION (OR RESIDUE METHOD)

Now by the definition of inverse z transform,

$$x(n)=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Using partial fraction expansion technique the function X (z) $z^{n-1}$ can be expressed as

$$X(z)\,z^{n-1}=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\frac{A_3}{z-p_3}+\ldots+\frac{A_N}{z-p_N}\text{------------------------------------} \textbf{(1)}$$

Where $p_1,p_2$ are the poles of X(z) and $A_1,A_2$ are the residues.

The residue $A_1$ is obtained by multiplying the equation by $(z-p_1)$ and letting $z=p_1$

Similarly the residues are evaluated.

$$A_1=(z-p_1)\,X(z)\,z^{n-1}|_{z=p1}$$

$$A_2=(z-p_2)\,X(z)\,z^{n-1}|_{z=p2}$$

$$A_3=(z-p_3)\,X(z)z^{n-1}|_{z=p3}$$

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$$A_N=(z-p_N)\,X(z)\,z^{n-1}|_{z=pN}$$

The equation (1) can be written as

$$x(n)=\frac{1}{2\pi j}\oint\left[\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\frac{A_3}{z-p_3}+\ldots+\frac{A_N}{z-p_N}\right]dz$$

Using Cauchy's integral theorem

$$x(n)=\sum_{i=1}^{N}[(z-p_{i)}X(z)z^{n-1}|_{z=pi}$$

### 2.2.2. PARTIAL FRACTION EXPANSION METHOD.

Let X (z) is Z transform of x (n).

$$X(z) = \frac{N(z)}{D(z)}$$

Where N (z) =Numerator polynomial of X (z)

D (z) = denominator polynomial of X (z)

$$\frac{X(z)}{z} = \frac{N(z)}{D(z)z}$$

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)}$$

On factorizing the denominator polynomial of equation

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)(z-p_2)(z-p_3)\dots\dots\dots(z-p_N)}$$

Where $p_1$, p2 are the poles of the denominator polynomial

Evaluation of residues:

The coefficients of the denominator polynomial D (z) are assumed real and so the roots of the denominator polynomial are real and or complex conjugate pairs. Hence on factorizing polynomial are poles of X (z).

**Case 1:** When the roots are real and distinct

In this case $\frac{X(z)}{z}$ *can be expressed as*

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)(z-p_2)(z-p_3)\dots\dots\dots(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \dots\dots\dots\dots\dots\dots\dots\dots + \frac{A_N}{z-p_N}$$

Where $A_1$, A2………………..$A_N$ are the residues

**Case 2**: When the roots have multiplicity

In this case $\frac{X(z)}{z}$ *can be expressed as*

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)(z-p_2)(z-p_3)\dots\dots\dots(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \dots\dots\dots\dots\dots\dots\dots\dots + \frac{A_N}{z-p_N}$$

The residue $A_{xr}$ of repeated root is obtained as shown below

$$A_{xr} = \frac{1}{r!}\frac{d^r}{dz^r}\left[(z-p_x)^q \frac{X(z)}{z}\right]\Big|_{z=p_x} \quad \text{where } r=0,1,2,\dots\dots\dots\dots\dots(q-1)$$

**Case 3**: When the roots are complex conjugate

In this case $\frac{X(z)}{z}$ *can be expressed as*

$$\frac{X(z)}{z}=\frac{Q(z)}{D(z)}=\frac{Q(z)}{(z-p_1)(z-p_2)(z-p_3)......(z^2+az+b)....(z-p_N)}$$

$$=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+......+\frac{A_x}{z-(x+iy)}+\frac{A_{x^*}}{z-(x-iy)}+.............+\frac{A_N}{z-p_N}$$

The residues of real and non repeated roots are evaluated as explained as case-1.

### 2.2.3. POWER SERIES EXPANSION METHOD

Let X (z) be z transform of x (n) and X(z) be a rational function of z as shown below

**Case-i:**

$$X(z)=\frac{N(z)}{D(z)}=c_0+c_1z^{-1}+c_2z^{-2}+c_3z^{-3}+.....\text{---------------------}\mathbf{(1)}$$

**Case-ii:**

$$X(z)=\frac{N(z)}{D(z)}=d_0+d_1z^{1}+d_2z^{2}+d_3z^{3}+.....\text{---------------------}\mathbf{(2)}$$

**Case-iii:**

$$X(z)=\frac{N(z)}{D(z)}=........+e_{-2}z^{2}+e_{-1}z^{1}+e_0+e_1z^{-1}+e_2z^{-2}+....\text{—}(3)$$

By the definition of Z transform, we get

$$X(n)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}$$

On expanding the summation we get

$$X(z)=..................x(-3)z^{3}+x(-2)z^{2}+x(-1)z^{1}+x(0)z^{0}+x(1)z^{-1}+.....\text{----}(4)$$

On comparing the coefficients of z of the equations 1 with 4 the samples are determined.