1.4 ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY STRAIN ENERGY METHOD

(PLANE FRAMES)

Let a statically indeterminate structure has degree of indeterminacy as *n* . On the selected basic determinate structure apply the unknown forces R_1, R_2..... and R_n . Using the Eq. (4.16) the displacement Δ_j in the direction of R_j is expressed by

$$\Delta_j = \frac{\partial U}{\partial R_j}$$

$$(j = 1, 2, \ldots \ldots n) \quad (5.1)$$

The equations (5.1) will provide the *n* linear simultaneous equations with *n* unknowns R_1, R_2 ..... and R_n. Since the Δ_j is known, therefore, the solution of simultaneous equations will provide the desired R_j (j =1, 2,…., n).

For structures with members subjected to the axial forces only (i.e. pin-jointed structures), the equation (5.1) is re-written as

$$\Delta_j = \frac{\partial}{\partial R_j}\sum\left(\frac{P^2 L}{2AE}\right) = \sum\left(\frac{P\dfrac{\partial P}{\partial R_j}L}{AE}\right) \quad (5.2)$$

Where;

P is the force in the member due to applied loading and unknown R_j (j =1, 2,…., n); and *L* and *AE* are length and axial rigidity of the member, respectively.

For structures with members subjected to the bending moments (i.e. beams and rigid-jointed frames), the equation (5.1) is re-written as

$$\Delta_j = \frac{\partial}{\partial R_j}\left(\int \frac{M^2 dx}{2EI}\right) = \int \frac{M\dfrac{\partial M}{\partial R_j}dx}{EI} \quad (5.3)$$

where;

M is the bending moment due to applied loading and unknown R_j (j =1, 2,…., n) at a small element of length dx ; and EI is the flexural rigidity.

Problem No:1

Determine the force in various members of the pin-jointed frame shown in Figure 5.20(a).Length and AE is constant for all members.

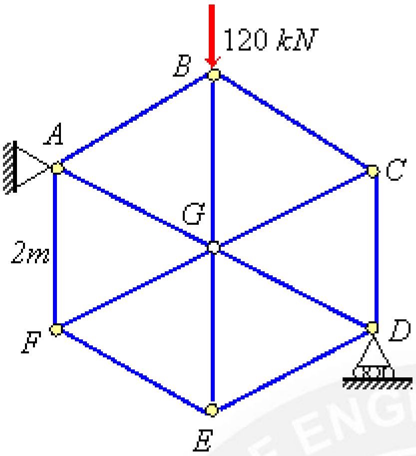


Figure 5.20(a)

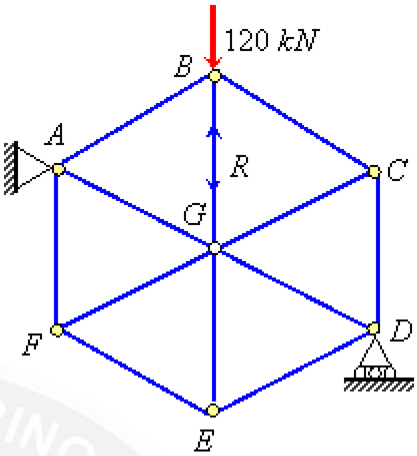


Figure 5.20(b)

Solution:

The static indeterminacy of the pin-jointed frame is =12+3–7×2 = 1. Let the force in the member BG be R as shown in Figure 5.20(b). According to the Castigliano's theorem

$$\frac{\partial U}{\partial R} = 0$$

The computation of $\partial U / \partial R$ is made in Table 5.6.

$$\frac{\partial U}{\partial R} = \frac{1}{AE}\left(P\frac{\partial P}{\partial R}L\right) = 0$$

$$\frac{1}{AE}(24R - 1920) = 0$$

The final force in various members of the frame is shown in Table 5.6

Member	Length,*L*(*m*)	*P*	$\frac{\partial P}{\partial R}$	$P\frac{\partial P}{\partial R}L$	Final force (kN)
AB	2	-120 +*R*	1	2 *R*-240	-40
AG	2	120 -*R*	-1	2 *R*-240	40
AF	2	-60 +*R*	1	2 *R*-120	20
BC	2	-120 +*R*	1	2 *R*-240	-40
BG	2	-*R*	-1	2 *R*	-80
CD	2	-120 +*R*	1	2 *R*-240	-40
CG	2	120 -*R*	-1	2 *R*-240	40
DE	2	-60 +*R*	-1	2 *R*-120	20
DG	2	60 -*R*	-1	2 *R*-120	-20
EF	2	-60 +*R*	1	2 *R*-120	20
EG	2	60 -*R*	-1	2 *R*-120	-20
FG	2	60 -*R*	-1	2 *R*-120	-20

$$\sum \quad 24R - 1920$$

$$\frac{\partial U}{\partial R} = \frac{1}{AE}\left(P\frac{\partial P}{\partial R}L\right) = 0$$

or

$$\frac{1}{AE}(24R - 1920) = 0$$

$$\therefore \quad R = 80 \text{ kN}$$

The final force in various members of the frame is shown in Table 5.6.

Problem No:2

Determine the force in various members of the pin-jointed frame as shown in Figure 5.21(a), if the member *BC* is short by an amount of Δ. All members of the frame have same axial rigidity as AE.

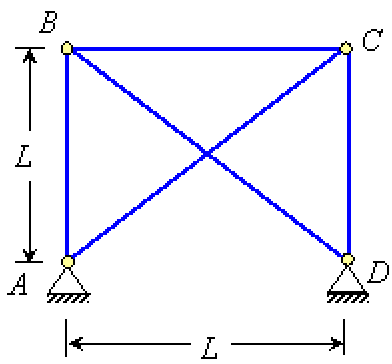


Figure 5.21(a)

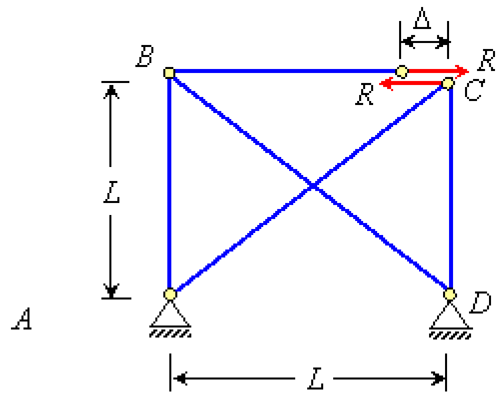


Figure 5.21(b)

Solution:

The static indeterminacy of the pin-jointed frame is =5 + 4 - 2×4=1. Since the member BC is short by an amount of Δ, therefore, apply a force R in the member BC such that displacement in the direction of R is Δ. Thus, according to the Castigliano's theorem.

$$\frac{\partial U}{\partial R} = \Delta$$

The computation of $\partial U / \partial R$ is made in Table 5.7.

Table 5.7

Member	Length, $L(m)$	F	$\frac{\partial F}{\partial R}$	$F \frac{\partial F}{\partial R} L$	Final force
AB	L	R	1	RL	1
AC	$\sqrt{2}L$	$-\sqrt{2}R$	$-\sqrt{2}$	$2\sqrt{2}RL$	$-\sqrt{2}$
BC	L	R	1	RL	1
BD	L	$-\sqrt{2}R$	$-\sqrt{2}$	$2\sqrt{2}RL$	$-\sqrt{2}$
CD	$\sqrt{2}L$	R	1	RL	1

$$\sum \left(\frac{AE\Delta}{(3+4\sqrt{2})L} \right)$$

$$(3+4\sqrt{2})RL$$

or
$$\frac{\partial U}{\partial R} = \frac{1}{AE}\left(F \frac{\partial F}{\partial R} L \right) = \Delta$$

$$\therefore \quad \frac{1}{AE}\left(3+4\sqrt{2}\right)RL = \Delta$$

$$R = \frac{AE\Delta}{(3+4\sqrt{2})L}$$

The final force in various members of the frame is shown in Table 5.7.

Problem No:3

Determine the horizontal reaction of the portal frame shown in Figure 5.22(a) by energy method. Also, calculate the horizontal reaction when the member BC is subjected to distributed load, *w* over entire length.

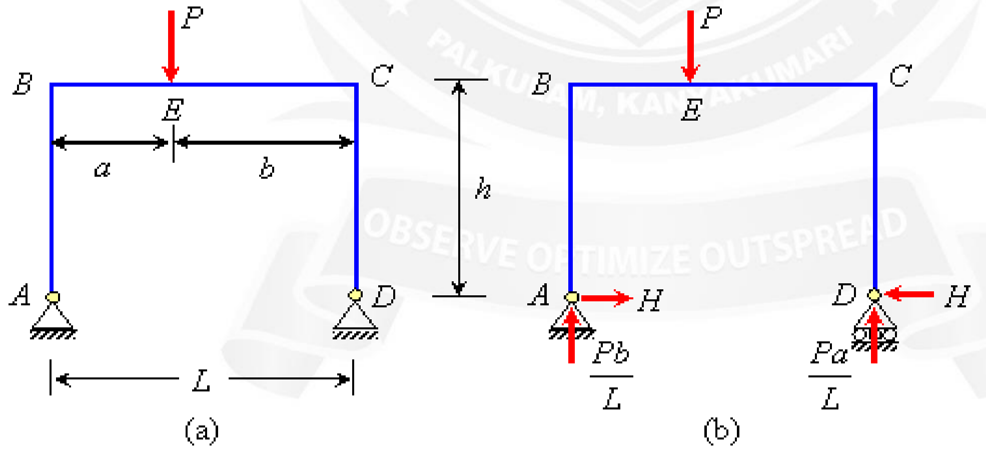


Figure 5.22(a)-(b)

Solution:

Static indeterminacy of the frame = 1.

Let the horizontal reaction, *H* at D be the redundant. The reaction at A and D are

$$H_A = H; \qquad H_D = H \qquad R_A = Pb/L \quad \text{and} \quad R_D = Pa/L$$

For the span AB (x measured from A),

$$M_x = Hx \qquad \text{and} \qquad \frac{\partial M_x}{\partial H} = x$$

$$\frac{\partial U_{AB}}{\partial H} = \frac{Hh^3}{3EI}$$

For the span BE (x measured from B),

$$M_x = Hh - \frac{Pb}{L}x \qquad \text{and} \qquad \frac{\partial M_x}{\partial H} = h$$

$$\frac{\partial U_{BE}}{\partial H} = \frac{1}{EI}\int_0^a \left[Hh - \frac{Pb}{L}x \right] h dx = \frac{Hh^2 a}{EI} - \frac{Pa^2 bh}{2EIL}$$

For the span CD (X Measured from D), $M_x = Hx \qquad \text{and} \qquad \frac{\partial M_x}{\partial H} = x$

$$\frac{\partial U_{CD}}{\partial H} = \frac{Hh^3}{3EI}$$

For the span CE (x measured from C),

$$M_x = Hh - \frac{Pa}{L}x \qquad \text{and} \qquad \frac{\partial M_x}{\partial H} = h$$

$$\frac{\partial U_{CE}}{\partial H} = \frac{Hh^2 b}{EI} - \frac{Pab^2 h}{2EIL}$$

Since $\frac{\partial U}{\partial H} = 0$ $\Rightarrow \frac{Hh^3}{3EI} + \frac{Hh^2 a}{EI} - \frac{Pa^2 bh}{2EIL} + \frac{Hh^2 b}{EI} - \frac{Pab^2 h}{2EIL} + \frac{Hh^3}{3EI}$

$$\frac{2Hh^2}{3} + HhL - \frac{Pab}{2} = 0$$

$$H = \frac{3Pab}{2h(2h + 3L)}$$

Horizontal reaction due to udl, w over BC :

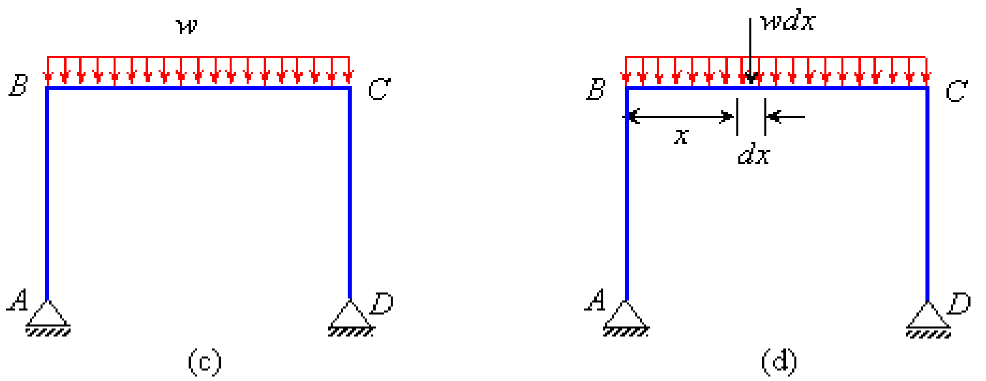


Figure 5.22(c)-(d)

The horizontal reaction due to small incremental load *wdx* is given by

$$dH = \frac{3wdx.x(L-x)}{2h(2h+3L)}$$

(using the expression derived earlier for concentrated force and putting

$P = wdx$, $a = x$ and $b = L - x$).

The horizontal reaction due to entire distributed load,

$$\begin{aligned} H &= \int dH \\ &= \int_0^L \frac{3wdx.x(L-x)}{2h(2h+3L)} \\ &= \frac{3w}{2h(2h+3L)} \int_0^L x(L-x)dx \\ &= \frac{wL^3}{4h(2h+3L)} \end{aligned}$$

Problem No:4

Analyze the portal frame shown in Figure 5.23 by strain energy method.

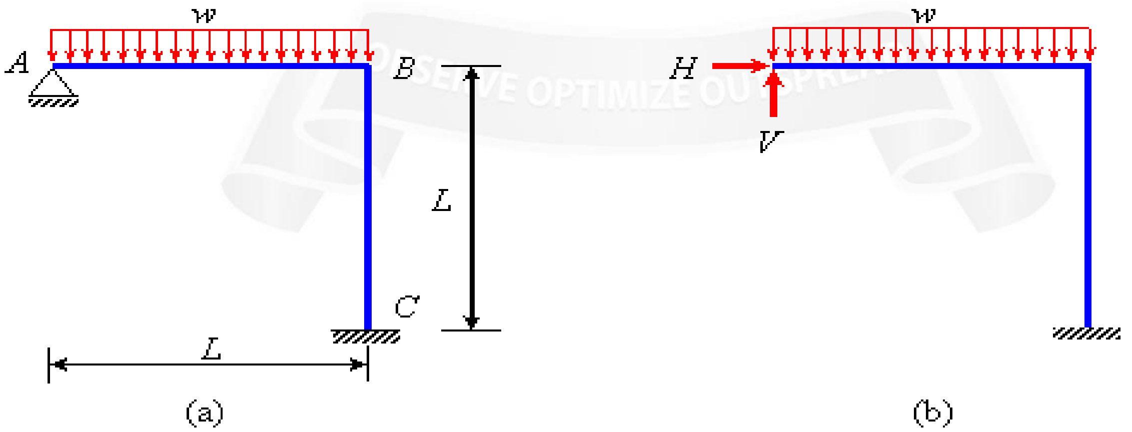


Figure 5.23

Solution:

Static indeterminacy of the frame = 2. Horizontal and vertical reactions at A are taken as redundant.

For the span *AB* (*x* measured from *A*),

$$M_x = Vx - wx^2/2$$

$$\frac{\partial M_x}{\partial H} = 0 \quad \text{and} \quad \frac{\partial M_x}{\partial V} = x$$

$$\frac{\partial U_{AB}}{\partial H} = 0$$

$$\frac{\partial U_{AB}}{\partial V} = \frac{1}{EI}\int_0^L \left[Vx - wx^2/2\right]\left[x\right]dx = \frac{VL^3}{3EI} - \frac{wL^4}{8EI}$$

$$M_x = Hx + VL - wL^2/2$$

$$\frac{\partial M_x}{\partial H} = x \quad \text{and} \quad \frac{\partial M_x}{\partial V} = L$$

$$\frac{\partial U_{BC}}{\partial H} = \frac{1}{EI}\int_0^L \left[Hx + VL - wL^2/2\right]\left[x\right]dx = \frac{VL^3}{2EI} + \frac{HL^3}{3EI} - \frac{wL^4}{4EI}$$

$$\frac{\partial U_{BC}}{\partial V} = \frac{1}{EI}\int_0^L \left[Hx + VL - wL^2/2\right]\left[L\right]dx = \frac{VL^3}{EI} + \frac{HL^3}{2EI} - \frac{wL^4}{2EI}$$

Since;

$$\frac{\partial U}{\partial H} = 0 \quad \Rightarrow \quad \frac{VL^3}{2EI} + \frac{HL^3}{3EI} - \frac{wL^4}{4EI} = 0 \quad \Rightarrow \quad \frac{VL^3}{2EI} + \frac{HL^3}{3EI} - \frac{wL^4}{4EI} = 0$$

or $6V + 4H = 3wL$ ------ (i)

and

$$\frac{\partial U}{\partial V} = 0 \quad \Rightarrow \quad \frac{VL^3}{3EI} - \frac{wL^4}{8EI} + \frac{VL^3}{EI} + \frac{HL^3}{2EI} - \frac{wL^4}{2EI} = 0$$

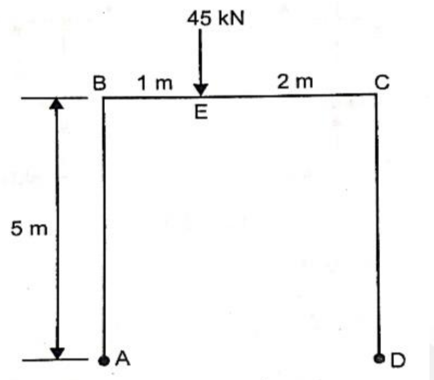
or $32V + 12H = 15Wl$ ------ (ii)

Solving Eqs (i) and (ii) for *H* and *V*,

$$H = \frac{3wL}{28} \quad \text{and} \quad V = \frac{3wL}{7}$$

Problem No:5

The simple portal frame shown in fig., is asymmetrically loaded.EI is constant.Analyse the frame by the strain energy method.Sketch the bending moment diagram.



Solution:

- **Finding the Redundant Force:**

  Degree of static indeterminacy = 1 x 3 – 2 = 1

  Let us treat the horizontal reaction at D as redundant.Since there is no other horizontal force,

  HA = - HD = H

  Since D is hinged,Δd = 0

  ∂U/∂H = 0

  $1/EI \frac{\int \; M \partial M}{\partial H} dx$ --- (1)

  VA x 3 – 45 x 2 = 0

  VA = 15 KN

Portion	Origin	Limits (m)	Mx (or) M	∂U/∂H
AB	A	0 to 5	- H.x	-x
BE	B	0 to 1	30x – H x 5	-5
CE	C	0 to 2	15x – H x 5	-5
DC	D	0 to 5	- H.x	-x

Substituting the values in equation (1)

$$\frac{1}{EI}\left\{\int_0^5 (-Hx)(-x)\,dx + \int_0^1 (30x - 5H)(-5)\,dx + \int_0^2 (15x - 5H)(-5)\,dx + \int_0^5 (-Hx)(-x)\,dx\right\} = 0$$

83.33 H – 75 + 25 H – 150 + 50 H = 0

158.33 H = 225

H = 1.421 KN

- **Determining the Bending Moments:**

  Span AB,
  x = 0,x = 5m,
  Mx = -Hx = -1.421x
  MA = - 7.11 kNm
  Span BE,
  x = 0,x = 1m,
  Mx = 30x - 5H
  MB = - 7.11 kNm
  ME = 22.89 kNm
  Span CE,
  x = 0,x = 2m,
  Mx = 15x - 5H
  MC = - 7.11 kNm
  ME = 22.89 kNm
  Span DC,
  x = 0,x = 5m,
  Mx = -Hx = -1.421x
  MD = 0 kNm
  ME = - 7.11 kNm
- **Bending Moments Diagram:**