2.3. SOLUTION OF DIFFERENCE EQUATIONS BY Z TRANSFORM

The one-sided z transform is a very efficient tool for the solution of difference equations with non zero initial conditions. It achieves that by reducing the difference equation relating the two time domain signals to an equivalent algebraic equation relating their one sided z transforms. This equation can be easily solved to obtain the transform of the desired signal. The signal in the time domain is obtained by inverting the resulting Z transform.

The one-sided or unilateral z transform of a signal x (n) is defined by

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Example-1:

Determine the step response of the system

$Y(n) = ay(n-1) + x(n) \quad -1<a<1$

When the initial condition is $y(-1) = 1$

Solution:

By taking the one-sided z transform of both sides we can

$Y(z) = a[z^{-1} Y(z) + y(-1)] + X(z)$

Upon substitution for y (-1) and X (z) and solving Y (z), we obtain the result

$$Y(z) = \frac{a}{1-az^{-1}} + \frac{1}{(1-az^{-1})(1-z^{-1})}$$

By performing a partial fraction expansion and inverse transforming the result we get

$$y(n) = a^{n+1} u(n) + \frac{1-a^{n+1}}{1-a} u(n)$$

$$y(n) = \frac{1}{1-a}(1-a^{n+2})u(n)$$

Example-2:

Determine the response of discrete time LTI system governed by the difference equation $y(n) = -0.8y(n-1) + x(n)$, when the input is unit step and initial condition,

a) $y(-1)=0$ and b) $y(-1)=2/9$

Solution:

Given that $x(n) = u(n)$;

$X(z) = Z\{x(n)\} = Z\{u(n)\} = z/z-1$

Given that, $y(n) = -0.8y(n-1) + x(n)$,

y (n)+ 0.8y(n-1)= x(n),

On taking the z transform of the above equation we get,

Y (z) +0.8[Z^{-1} Y (z) + y (-1)] =X (z)

Y (z) [1+0.8 Z^{-1}+0.8 y (-1) = $\frac{z}{z-1}$

Y (z) (1+$\frac{0.8}{z}$) = $\frac{z}{z-1} - 0.8$ y(-1)

Y (z) ($\frac{z+0.8}{z}$) = $\frac{z}{z-1} - 0.8$ y(-1)

Y (z) = $\frac{z^2}{(z-1)(z+0.8)}$ - 0.8 $\frac{zy(-1)}{(z+0.8)}$

Let P (z) = $\frac{z^2}{(z-1)(z+0.8)}$ => $\frac{P(z)}{z} = \frac{z}{(z-1)(z+0.8)}$

Let $\frac{z}{(z-1)(z+0.8)} = \frac{A}{z-1} + \frac{B}{z+0.8}$

A= $\frac{z}{(z-1)(z+0.8)}(z-1)$ | $_{z=1}$

= $\frac{1}{1+0.8}$

=$\frac{1}{1.8}$

=$\frac{10}{18}$

=5/9

B = $\frac{z}{(z-1)(z+0.8)}(z+0.8)$ | $_{z=-0.8}$

=$-\frac{0.8}{-0.8-1}$

=$\frac{0.8}{1.8}$

=$\frac{8}{18}$

=$\frac{4}{9}$

$\frac{P(z)}{z} = \frac{z}{(z-1)(z+0.8)}$

= $\frac{5}{9}\frac{1}{z-1} + \frac{4}{9}\frac{1}{z+0.8}$

Y (z) = $\frac{5}{9}\frac{z}{z-1} + \frac{4}{9}\frac{z}{z+0.8}$ - 0.8 $\frac{zy(-1)}{(z+0.8)}$

a) When y(-1)=0

Y (z) = $\frac{5}{9}\frac{z}{z-1} + \frac{4}{9}\frac{z}{z+0.8}$

Response (n) $= z^{-1}\{Y(z)\} = z^{-1}\{\frac{5}{9}\frac{z}{z-1} + \frac{4}{9}\frac{z}{z+0.8}\}$

$= \frac{5}{9}u(n) + \frac{4}{9}(-0.8)^n u(n)$

b) When y (-1) = 2/9

When y (-1) = 2/9, we get

$Y(z) = \frac{5}{9}\frac{z}{z-1} + \frac{4}{9}\frac{z}{z+0.8} - 0.8\frac{zy(-1)}{(z+0.8)}$

$= \frac{5}{9}\frac{z}{z-1} + \frac{4}{9}\frac{z}{z+0.8} - 0.8 \times \frac{2}{9}\frac{z}{z+0.8}$

$= \frac{5}{9}\frac{z}{z-1} + \frac{2.4}{9}\frac{z}{z+0.8}$

$= \frac{5}{9}\frac{z}{z-1} + \frac{24}{90}\frac{z}{z+0.8}$

$= \frac{5}{9}\frac{z}{z-1} + \frac{12}{45}\frac{z}{z+0.8}$

Response (n) $= z^{-1}\{Y(z)\} = z^{-1}\{\frac{5}{9}\frac{z}{z-1} + \frac{12}{45}\frac{z}{z+0.8}\}$

$= [\frac{5}{9} + \frac{12}{45}(-0.8)^n] u(n)$

Response $= [\frac{5}{9} + \frac{12}{45}(-0.8)^n] u(n)$