Volume integral

An integral which is evaluated over a volume bounded by a surface is called a volume integral.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ is a vector field in V, then the volume integral is defined by

$$\iiint_V \vec{F}\ dv$$

Example: If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\ dv$ where v is the volume of the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

Given $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-4+3) + \vec{k}(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

For limits

Given $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$

$\therefore z : 0 \to 4 - 2x - 2y$

Put $z = 0 \Rightarrow 2x + 2y = 4\ (or)\ x + y = 4$

$\therefore y : 0 \to 2 - x$

Put $z = 0, y = 0 \Rightarrow 2x = 4\ (or)\ x = 2$

$\therefore x : 0 \to 2$

$$\therefore \int\int\int \nabla \times \vec{F}\ dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})dzdydx$$

$$= \int_0^2 \int_0^{2-x} (\vec{j} - 2y\vec{k})[z]_0^{4-2x-2y}\ dydx$$

$$= \int_0^2 \int_0^{2-x} [(4-2x-2y)\vec{j} - 2y(4-2x-2y)\vec{k}]\ dydx$$

$$= \int_0^2 \left\{\left[4y - 2xy - \frac{2y^2}{2}\right]\vec{j} - \left[4y^2 - 2xy^2 - \frac{4y^3}{3}\right]\vec{k}\right\}_0^{2-x} dx$$

$$= \int_0^2 \{[4(2-x) - 2x(2-x) - (2-x)^2]\,\vec{j} - \left[4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\vec{k}\right]\} dx$$

$= \int_0^2 [8 - 4x - 4x + 2x^2 - 4 + 4x - x^2]\,\vec{j} -$

$\left[16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3 - \frac{4}{3}\,(8 - 12x + 6x^2 - x^3)\,\vec{k}\right] dx$

$= \int_0^2 \left[(4 - 4x + x^2)\,\vec{j} - \frac{\vec{k}}{3}\,(16 - 24x + 12x^2 - 2x^3)\right] dx$

$= \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 \vec{j} + \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$

$= \left(8 - 8 + \frac{8}{3}\right)\vec{j} - \frac{\vec{k}}{3}\,(32 - 48 + 32 - 8)$

$= \frac{8}{3}\,(\vec{j} - \vec{k})$