Line Integral over a plane curve

An integral which is evaluated along a curve then it is called line integral.

Let C be the curve in same region of space described by a vector valued function

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ of a point (x, y, z) and let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuous vector valued function defined along a curve C. Then the line integral $\vec{F}$ over C is denoted by

$\int_c \vec{F} \cdot d\vec{r}$.

Work done by a Force

If $\vec{F}(x, y, z)$ is a force acting on a particle which moves along a given curve C, then

$\int_c \vec{F} \cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in the displacement along C.

Thus work done by force $\vec{F} = \int_c \vec{F} \cdot d\vec{r}$

Conservative force field

The line integral $\int_A^B \vec{F} \cdot d\vec{r}$ depends not only on the path C but also on the end points A and B.

If the integral depends only on the end points but not on the path C, then $\vec{F}$ is said to be conservative vector field.

If $\vec{F}$ is conservative force field, then it can be expressed as the gradient of some scalar function φ.

$(ie)\ \vec{F} = \nabla\varphi$

$$\vec{F} = \nabla\varphi = \left(\vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}\right)$$

$$\vec{F} \cdot d\vec{r} = \left(\vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}\right) \cdot \left(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right)$$

$$= \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy + \frac{\partial\varphi}{\partial z}\,dz = \partial\varphi$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_A^B \partial\varphi$$

$$= [\varphi]_A^B$$

$$= \varphi[B] - \varphi[A]$$

$\therefore$ work done by $\vec{F} = \varphi[B] - \varphi[A]$

Note:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \nabla \times (\nabla\varphi) = \vec{0}$ and hence $\vec{F}$ is irrotational.

Example: If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution:

Given $\vec{F} = 3xy\vec{i} - y^2\vec{j}$

$d\vec{r} = dx\vec{i} + dy\,\vec{j}$

$\vec{F} \cdot d\vec{r} = 3xy\,dx - y^2 dy$

Given C is $y = 2x^2$

$\therefore dy = 4xdx$

Along C, x varies from 0 to 1.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 3x\,(2x^2)dx - 4x^4(4xdx)$$

$$= \int_0^1 6x^3 - 16x^5\, dx$$

$$= \left[6\frac{x^4}{4} - 16\,\frac{x^6}{6}\right]$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6} \text{ units.}$$

Example: Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given $y^2 = x \Rightarrow 2ydy = dx$

Along the curve C, y varies from 0 to 1.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 ((y^2)^2 - y^2 + y^2)\,2ydy - (2(y^2)y + y)dy$$

$$= \int_0^1 (2y^5 - 2y^3 + 2y^3 - 2y^3 - y)\,dy$$

$$= \int_0^1 (2y^5 - 2y^3 - y)\,dy$$

$$= \left[2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{2}{6} - \frac{2}{4} - \frac{1}{2} = -\frac{2}{3}$$

Example: Find the work done in moving a particle in the force field

$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ **from** $t = 0$ **to** $t = 1$ **along the curve** $x = 2t^2, y = t, z = 4t^3$.

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 3x^2dx + (2xz - y)dy - zdz$$

Given $x = 2t^2$, $y = t$, $z = 4t^3$

$$dx = 4tdt,\quad dy = dt,\quad dz = 12t^2dt$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 48t^5dt + (16t^5 - t)dt - 48t^5\,dt$$

$$= \int_0^1 (16t^5 - t)dt$$

$$= \left[\frac{16\,t^6}{6} - \frac{t^2}{2}\right]_0^1 = \frac{16}{6} - \frac{1}{2} = \frac{13}{6}$$

Example: If $\vec{F} = (3x^2 + 6y)\vec{i} + 14yz\vec{j} + 20xz^2\vec{k}$**, evaluate** $\int_c \vec{F} \cdot d\vec{r}$ **from** $(0,0,0)$ **to** $(1,1,1)$ **along the curve** $x = t,\ y = t^2,\ z = t^3$.

Solution:

Given $\vec{F} = (3x^2 + 6y)\vec{i} + 14yz\vec{j} + 20xz^2\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (3x^2 + 6y)dx + 14yzdy + 20xz^2dz$$

Given $x = t$, $y = t^2$, $z = t^3$

$$dx = dt,\quad dy = 2tdt,\quad dz = 3t^2dt$$

The point $(0,0,0)$ to $(1,1,1)$ on the curve correspond to $t = 0$ and $t = 1$.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 6t^2)dt + 14t^5(2t\,dt) + 20t^7(3t^2)dt$$

$$= \int_0^1 (9t^2 + 28\,t^6 + 60t^9)\,dt$$

$$= \left[9\frac{t^3}{3} + 28\frac{t^7}{7} + 60\,\frac{t^9}{9}\right]_0^1$$

$$= \frac{9}{3} + \frac{28}{7} + \frac{60}{10} = 3 + 4 + 6 = 13\text{units.}$$

Example: Find $\int_c \vec{F} \cdot d\vec{r}$ where c is the circle $x^2 + y^2 = 4$ in the xy plane where $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$.

Solution:

Given $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

In xy plane $z = 0 \Rightarrow dz = 0$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 2xydx + x^2dy$$

Given C is $x^2 + y^2 = 4$

The parametric form of circle is

$$x = 2\cos\theta, \qquad y = 2\sin\theta$$

$$dx = -2\sin\theta d\theta, \quad dy = 2\cos\theta d\theta$$

And θ varies from 0 to 2π

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [2(2\cos\theta)(2\sin\theta)]\,(-2\sin\theta d\theta) + (2\cos\theta)^2 2\cos\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta sin^2\theta + 8\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta(1 - cos^2\theta) + 8\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta + 16\, cos^3\theta + 8\cos^3\theta\, d\theta$$

$$= -16\int_0^{2\pi} \cos\theta\, d\theta + 24\int_0^{2\pi} \cos^3\theta\, d\theta$$

$$= -16\int_0^{2\pi} \cos\theta\, d\theta + 24\int_0^{2\pi} \frac{3\cos\theta + \cos 3\theta}{4} d\theta$$

$$= 16\,[\sin\theta]_0^{2\pi} + \frac{24}{4}\left[3\sin\theta + \frac{sin3\theta}{3}\right]_0^{2\pi}$$

$$= 0 \qquad [\because \sin n\pi = 0, sin0 = 0]$$

Example: State the physical interpretation of the line integral $\int_A^B \vec{F} \cdot d\vec{r}$.

Solution:

Physically $\int_A^B \vec{F} \cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, displacing a particle from A to B along the curve C.

Example: If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$, check whether the integral

$\int_c \vec{F} \cdot d\vec{r}$ **is independent of the path C.**

Solution:

Given $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (4xy - 3x^2z^2)dx + 2x^2dy - 2x^2zdz$$

Then $\int_c \vec{F} \cdot d\vec{r}$ is independent of path C if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-6x^2z + 6x^2z) + \vec{k}(4x - 4x)$$

$$= \vec{0}$$

Hence the line integral is independent of path.

Example: Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

Solution:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \vec{0}$.

$$\text{Now, } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is a conservative vector field.