Two Dimensional Heat Equation

The 2-D heat equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The various possible solution of 2-D heat equation is

i) u(x ,y)= (A cospx + Bsinpx) (C e ^{py} *+ e* $^{-py}$*)*

ii) u(x ,y) = (Ae^{px} *+ Be* $^{-px}$*) (C cospy + sin py)*

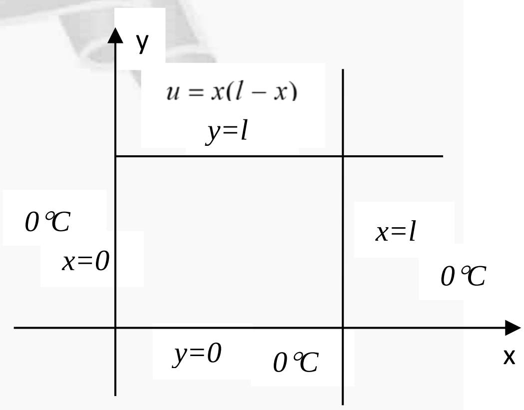
iii) u(x ,y) = (Ax+B) (Cy+D)

1. A square plate is bounded by the lines $x = 0, x = l, y = 0$ and $y = l$ **, its faces are insulated. The temperature along upper horizontal edge is given by** $u = x(l - x)$ when $0 < x < l$. **while the other three edges are kept at** $0°C$ **. Find steady state solution in the plate.**

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



The Boundary conditions are

$i)\ u(0, y) = 0$

$ii)\ u(l, y) = 0$

$iii)\ u(x, 0) = 0$

$iv)\ u(x,l) = f(x) = x(l-x), \quad 0 \le x \le l$

Here the non zero temperature is parallel to x axis then the

Correct solution is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \text{ -----(1)}$$

Apply condition (i) in (1)

$$u(0,y) = (A\cos 0 + B\sin 0)(Ce^{py} + De^{-py})$$

$$0 = A(Ce^{py} + De^{-py})$$

Here $Ce^{py} + De^{-py} \neq 0 \ \therefore A = 0$

Sub A in (1)

$$u(x,y) = (B\sin px)(Ce^{py} + De^{-py}) \text{ -----(2)}$$

Apply condition (ii) in (2)

$$u(l,y) = (B\sin pl)(Ce^{py} + De^{-py})$$

$$0 = (B\sin pl)(Ce^{py} + De^{-py})$$

Here $Ce^{py} + De^{-py} \neq 0,\ B \neq 0 \ \therefore \sin pl = 0$

$$\sin pl = \sin n\pi \ \Rightarrow pl = n\pi \ \Rightarrow p = \frac{n\pi}{l}$$

Sub p in (2)

$$u(x,y)=\left(B\sin\frac{n\pi x}{l}\right)\left(Ce^{\frac{n\pi y}{l}}+De^{-\frac{n\pi y}{l}}\right) -----(3)$$

Apply condition (iii) in (3)

$$u(x,0)=\left(B\sin\frac{n\pi x}{l}\right)\left(Ce^{0}+De^{-0}\right)$$

$$0=\left(B\sin\frac{n\pi x}{l}\right)\left(C+D\right)$$

Here $\sin\frac{n\pi x}{l}\neq 0, B\neq 0, \ \therefore C+D=0 \ \Rightarrow D=-C$

Sub $D=-C$ in (3)

$$u(x,y)=\left(B\sin\frac{n\pi x}{l}\right)\left(Ce^{\frac{n\pi y}{l}}-Ce^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y)=BC\sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}}-e^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y)=BC\sin\frac{n\pi x}{l}\left(2\sinh\frac{n\pi y}{l}\right) \because \frac{e^{\theta}-e^{-\theta}}{2}=\sinh\theta$$

$$u(x,y)=b_1\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \qquad let \ \ 2BC=b_1$$

The most general Solution is

$$u(x,y)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \quad -----(4)$$

Apply condition (iv) in (4)

$$u(x,l)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\sinh n\pi$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \qquad let\ B_n=b_n\sinh n\pi$$

This is Fourier sine series in*(0,l)*

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[\left(lx-x^2\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\left(l-2x\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+\left(-2\right)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{-4l^3}{ln^3\pi^3}\left[\cos n\pi-\cos 0\right]$$

$$b_n\sinh n\pi=\frac{-4l^2}{n^3\pi^3}\left[(-1)^n-1\right]$$

$$b_n=\frac{-4l^2}{n^3\pi^3\sinh n\pi}\left[(-1)^n-1\right]$$

$$b_n=\begin{cases}0 & if\ n\ is\ even\\ \frac{8l^2}{n^3\pi^3\sinh n\pi} & if\ n\ is\ odd\end{cases}$$

Sub b_n in (4)

$$u(x,y)=\sum_{n=1,3,5...}^{\infty}\frac{8l^2}{n^3\pi^3\sinh n\pi}\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

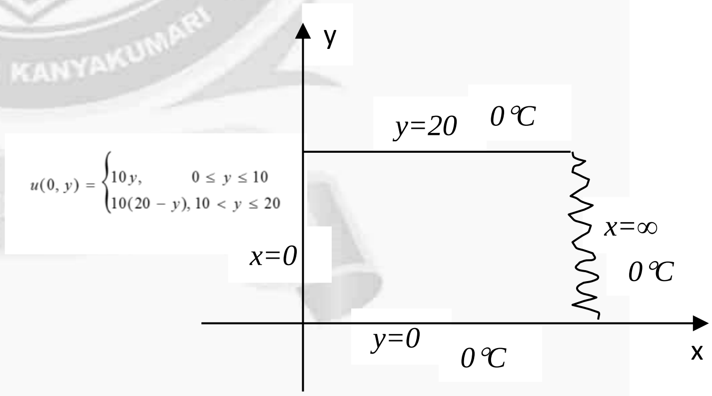
$$u(x,y)=\frac{8l^2}{\pi^3}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^3\sinh n\pi}\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

2. **A rectangular plate with insulated surface is 20cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge** $x=0$ **is given by** $u=\begin{cases}10y, & 0\le y\le 10\\ 10(20-y), & 10\le y\le 20\end{cases}$ **and the two long edges as well as the other short edges are kept at** $0^\circ C$**. Find the steady state temperature distribution in the plate.**

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



The Boundary conditions are

$i)\ u(x,0)=0$

$ii)\ u(x,20)=0$

$iii)\ u(\infty,y)=0$

$iv)\ u(0,y)=\begin{cases}10y, & 0\le y\le 10\\ 10(20-y), & 10< y\le 20\end{cases}$

The correct solution is

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos py+D\sin py\right) -----(1)$$

Apply condition (i) in (1)

$$u(x,0)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos 0+D\sin 0\right)$$

$$0=\left(Ae^{px}+Be^{-px}\right)C$$

Here $\left(Ae^{px}+Be^{-px}\right)\neq 0,\ \boxed{\therefore C=0}$

Sub C in (1)

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin py\right) -----(2)$$

Apply condition (ii) in (2)

$$u(x,20)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin 20p\right)$$

$$0=\left(Ae^{px}+Be^{-px}\right)\left(D\sin 20p\right)$$

Here $\left(Ae^{px}+Be^{-px}\right)\neq 0,\ D\neq 0$

$$\therefore \sin 20p=0 \Rightarrow \sin 20p=\sin n\pi \quad \Rightarrow 20p=n\pi \quad \Rightarrow \boxed{p=\frac{n\pi}{20}}$$

Sub p in (2)

$$(2)\ \Rightarrow u(x,y)=\left(Ae^{\frac{n\pi x}{20}}+Be^{-\frac{n\pi x}{20}}\right)\left(D\sin\frac{n\pi y}{20}\right) -----(3)$$

Apply condition (iii) in (3)

$$u(\infty, y) = \left(Ae^{\infty} + Be^{-\infty}\right)\left(D\sin\frac{n\pi y}{20}\right)$$

$$0 = \left(Ae^{\infty}\right)\left(D\sin\frac{n\pi y}{20}\right)$$

temperature $e^{\infty} \neq 0, D \neq 0 ,\sin\frac{n\pi y}{20} \neq 0, \therefore \boxed{A = 0}$

sub A in (3)

$$(3) \Rightarrow u(x, y) = \left(Be^{-\frac{n\pi x}{20}}\right)\left(D\sin\frac{n\pi y}{20}\right)$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20} e^{-\frac{n\pi x}{10}} \text{---------}(4)$$

Apply condition (iv) in (4)

$$u(0, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20} e^{-0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20}$$

This is half range sine series in (0,20)

$$b_n = \frac{2}{l}\int_0^l f(y)s\sin\frac{n\pi y}{l}dy$$

$$= \frac{2}{20}\int_0^{20} f(y)\sin\frac{n\pi y}{20}dy$$

$$= \frac{1}{10}\left[\int_0^{10} 10y\sin\frac{n\pi y}{20}dy + \int_{10}^{20} 10(20-y)\sin\frac{n\pi y}{20}dy\right]$$

$$=\frac{10}{10}\left[\int_0^{10} y\sin\frac{n\pi y}{20}dy+\int_{10}^{20}(20-y)\sin\frac{n\pi y}{20}dy\right]$$

$$=\left[(y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right)-(1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{400}}\right)\right]_0^{10}+\left[(20-y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right)-(-1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{400}}\right)\right]_{10}^{20}$$

$$=\left[-\frac{20}{n\pi}(y)\cos\frac{n\pi y}{20}+\frac{400}{n^2\pi^2}\sin\frac{n\pi y}{20}\right]_0^{10}+\left[-\frac{20}{n\pi}(20-y)\cos\frac{n\pi y}{20}-\frac{400}{n^2\pi^2}\sin\frac{n\pi y}{20}\right]_{10}^{20}$$

$$=\left[\left(-\frac{20}{n\pi}(10)\cos\frac{n\pi}{2}+\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)-(0)\right]+\left[(0)-\left(-\frac{20}{n\pi}(10)\cos\frac{n\pi}{2}-\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

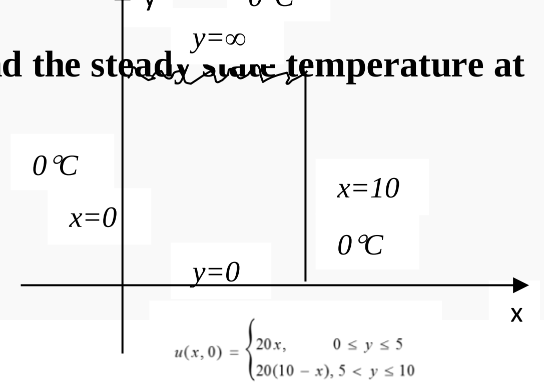
$$=-\frac{200}{n\pi}\cos\frac{n\pi}{2}+\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{200}{n\pi}\cos\frac{n\pi}{2}+\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\boxed{b_n=\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

Sub b_n in (4)

$$u(x,y)=\sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{20}e^{-\frac{n\pi x}{10}} \qquad \boxed{u(x,y)=\frac{800}{n^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{20}e^{-\frac{n\pi x}{10}}}$$

3. An infinitely long rectangular plate is of width 10cm. The temperature along the short edge *y=0* is given by $u=\begin{cases}20x, & 0\le x\le 5\\ 20(10-x), & 5\le x\le 10\end{cases}$**. If all the other edges are kept at zero temperature. Find the steady state temperature at any point on it.**



Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

The Boundary conditions are

$i)\ u(0,y)=0$

$ii)\ u(10,y)=0$

$iii)\ u(x,\infty)=0$

$$iv)\ u(x,0)=\begin{cases}20x, & 0\le x\le 5\\ 20(10-x), & 5<x\le 10\end{cases}$$

Here the non zero boundary condition is parallel to x axis then

The correct solution is

$$u(x,y)=\left(A\cos px+B\sin px\right)\left(Ce^{py}+De^{-py}\right)-----(1)$$

Apply condition (i) in (1)

$$u(0,y)=\left(A\cos 0+B\sin 0\right)\left(Ce^{py}+De^{-py}\right)$$

$$0=A\left(Ce^{py}+De^{-py}\right)$$

Here $\left(Ce^{py}+De^{-py}\right)\neq 0,\ \therefore \boxed{A=0}$

Sub C in (1)

$$u(x,y)=\left(B\sin px\right)\left(Ce^{py}+De^{-py}\right)-----(2)$$

Apply condition (ii) in (2)

$$u(10, y) = \left(B\sin 10p\right)\left(Ce^{py} + De^{-py}\right)$$

$$0 = \left(B\sin 10p\right)\left(Ce^{py} + De^{-py}\right)$$

Here $B \neq 0, \left(Ce^{py} + De^{-py}\right) \neq 0 \ \therefore \sin 10p = 0$

$$\therefore \sin 10p = 0 \Rightarrow \sin 10p = \sin n\pi \quad \Rightarrow 10p = n\pi \quad \Rightarrow \boxed{p = \frac{n\pi}{10}}$$

Sub p in (2)

$$u(x, y) = \left(B\sin\frac{n\pi x}{10}\right)\left(Ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) - - - - - (3)$$

Apply condition (iii) in (3)

$$u(x, \infty) = \left(B\sin\frac{n\pi x}{10}\right)\left(Ce^{\infty} + De^{-\infty}\right)$$

$$0 = \left(B\sin\frac{n\pi x}{10}\right)\left(Ce^{\infty} + De^{-\infty}\right)$$

Here $B \neq 0, e^{\infty} \neq 0, \sin\frac{n\pi x}{10} \neq 0 \ \therefore C = 0$

sub C in (3)

$$u(x, y) = \left(B\sin\frac{n\pi x}{10}\right)\left(De^{-\frac{n\pi y}{10}}\right)$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \ - - - - - (4)$$

Apply condition (iv) in (4)

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-0}$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

This is half range sine series in (0,10)

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx$$

$$=\frac{1}{10}\left[\int_0^5 20x\sin\frac{n\pi x}{20}dx+\int_5^{10} 20(10-x)\sin\frac{n\pi x}{20}dx\right]$$

$$=\frac{20}{10}\left[\int_0^5 x\sin\frac{n\pi x}{10}dx+\int_5^{10}(10-x)\sin\frac{n\pi x}{10}dx\right]$$

$$=2\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^5+\left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_5^{10}\right\}$$

$$=2\left\{\left[-\frac{10}{n\pi}(x)\cos\frac{n\pi x}{10}+\frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^5+\left[-\frac{10}{n\pi}(10-x)\cos\frac{n\pi x}{10}-\frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_5^{10}\right\}$$

$$=2\left\{\left[\left(-\frac{10}{n\pi}(5)\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)-(0)\right]+\left[(0)-\left(-\frac{10}{n\pi}(5)\cos\frac{n\pi}{2}-\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$=2\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n=\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Sub b_n in (4)

$$u(x,y)=\sum_{n=1}^{\infty}\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}} \Rightarrow \boxed{u(x,y)=\frac{400}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}}$$

4. **An infinite long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and one short edge are kept at 0°C, while the other short edge x = 0 is kept at temperature** $u=\begin{cases}20y, & 0\le y\le 5\\ 20(10-y), & 5<y\le 10\end{cases}$**. Find the steady state temperature distribution in the plate.**

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

The Boundary conditions are

$i)\ u(x,0)=0$

$ii)\ u(x,10)=0$

$iii)\ u(\infty,y)=0$

$$iv)\ u(0,y)=\begin{cases}20y, & 0\le y\le 5\\ 20(10-y), & 5<y\le 10\end{cases}$$

The correct solution is

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos py+D\sin py\right) -----(1)$$

Apply condition (i) in (1)

$$u(x,0) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos 0 + D\sin 0\right)$$

$$0 = \left(Ae^{px} + Be^{-px}\right)C$$

Here $\left(Ae^{px} + Be^{-px}\right) \neq 0, \ \therefore C = 0$

Sub C in (1)

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin py\right) -----(2)$$

Apply condition (ii) in (2)

$$u(x,10) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin 10p\right)$$

$$0 = \left(Ae^{px} + Be^{-px}\right)\left(D\sin 10p\right)$$

Here $\left(Ae^{px} + Be^{-px}\right) \neq 0, \ D \neq 0$

$$\therefore \sin 10p = 0 \Rightarrow s\,\text{in}\,10p = 0\sin n\pi \quad \Rightarrow 10p = n\pi \quad \Rightarrow p = \frac{n\pi}{10}$$

Sub p in (2)

$$(2) \ \Rightarrow u(x,y) = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right)\left(D\sin\frac{n\pi y}{10}\right) -----(3)$$

Apply condition (iii) in (3)

$$u(\infty,y) = \left(Ae^{\infty} + Be^{-\infty}\right)\left(D\sin\frac{n\pi y}{10}\right)$$

$$0 = \left(Ae^{\infty}\right)\left(D\sin\frac{n\pi y}{10}\right)$$

temperature $e^{\infty} \neq 0, D \neq 0$, $\sin\frac{n\pi y}{10} \neq 0, \therefore A = 0$

sub A in (3)

$$(3) \Rightarrow u(x,y) = \left(Be^{-\frac{n\pi x}{10}}\right)\left(D\sin\frac{n\pi y}{10}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{10} e^{-\frac{n\pi x}{10}} --------(4)$$

Apply condition (iv) in (4)

$$u(0,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{10} e^{-0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{10}$$

Which is half range sine series in (0,10)

$$b_n = \frac{2}{l}\int_0^l f(y)s\sin\frac{n\pi y}{10}dy$$

$$b_n = \frac{2}{10}\int_0^{10} f(y)s\sin\frac{n\pi y}{10}dy$$

$$b_n = \frac{2}{10}\left[\int_0^5 20y\sin\frac{n\pi y}{10}dy + \int_5^{10} 20(10-y)\sin\frac{n\pi y}{10}dy\right]$$

$$b_n = \frac{40}{10}\left[\int_0^5 y\sin\frac{n\pi y}{10}dy + \int_5^{10} (10-y)\sin\frac{n\pi y}{10}dy\right]$$

$$= \frac{20}{5}\left[\left(y\left(\frac{-\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right)-(1)\left(\frac{-\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right)\right)_0^5+\left((10-x)\left(\frac{-\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right)-(-1)\left(\frac{-\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right)\right)_5^{10}\right]$$

$$b_n = 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Sub b_n in (4)

$$u(x,y)=\sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\ \sin\frac{n\pi y}{10}\,e^{-\frac{n\pi x}{10}}$$

$$u(x,y)=\frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\ \sin\frac{n\pi y}{10}\,e^{-\frac{n\pi x}{10}}$$