DIVERGENCE AND CURL

Divergence of a vector function

If $\vec{F}(x, y, z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\vec{F}$ is defined by

$$\nabla.\vec{F} = \text{div } \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \text{ where } \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Note: $\nabla.\vec{F}$ Is a scalar point function.

Curl of a vector function

If $\vec{F}(x, y, z)$ is a differentiable vector point function defines at each point (x, y, z) in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Note: $\nabla \times \vec{F}$ Is a vector point function.

Example: If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$ at the point (1,-1, 1).

Solution:

Given $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

(i) $\nabla.\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$

$= y^2 + 2x^2z - 6yz$

$\nabla.\vec{F}_{(1,-1,1)} = 1 + 2 + 6 = 9$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & 3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y} - \frac{\partial(2x^2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(-3yz^2)}{\partial x} - \frac{\partial(xy^2)}{\partial z}\right] + \vec{k}\left[\frac{\partial(2x^2yz)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{\imath}(-3+2) + \vec{k}(-4+2)$$
$$= -\vec{\imath} - 2\vec{k}$$

Example: If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$, $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$$= (2x + 2z) + (-x + z) + 2z$$
$$= x + 5z$$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$$

$$= -(x+y)\vec{\imath} - (2x - 2x)\vec{\jmath} + (y+z)\vec{k}$$

$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{\imath} + 2\vec{k}$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$$= \vec{\imath} + 5\vec{k}$$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{\imath} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x+y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y+z)$

$$= -1 + 0 + 1$$

$\nabla \cdot (\nabla \times \vec{F})_{(1,1,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x+y) & 0 & y+z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{\imath} + \vec{k}$

Example: Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Solution:

Given $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$$= \vec{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+\vec{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)+\vec{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\vec{F} = \vec{i}(3x^2-3yz)+\vec{j}(3y^2-3xz)+\vec{k}(3z^2-3xy)$$

Now $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$

$$= 6x+6y+6z$$

$$= 6(x+y+z)$$

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \vec{i}[-3x+3x]-\vec{j}[-3y+3y]+\vec{k}[-3z+3z]$$

$$= \vec{0}$$

Example: Find div(grad φ) and curl(grad φ) at (1,1,1) for $\varphi = x^2y^3z^4$

Solution:

Given $\varphi = x^2y^3z^4$

$$\text{grad}\,\varphi = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4)+\vec{j}(x^23y^2z^4)+\vec{k}(x^2y^34z^3)$$

$$\text{Div}(\text{grad}\,\varphi) = \nabla\cdot(\text{grad}\,\varphi)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4)+\frac{\partial}{\partial y}(x^23y^2z^4)+\frac{\partial}{\partial z}(x^2y^34z^3)$$

$$= 2y^3z^4+6x^2yz^4+12x^2y^3z^4$$

$$\therefore \text{Div}(\text{grad}\,\varphi)_{(1,1,1)} = 2+6+12 = 20$$

$$\text{Curl}(\text{grad}\,\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^23y^2z^4 & x^2y^34z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3-12x^2y^2z^3)-\vec{j}(8xy^3z^3-8xy^3z^3)+\vec{k}(6xy^2z^4-6xy^2z^4)$$

$$= \vec{0}$$

$$\therefore \text{Curl grad}\varphi_{(1,1,1)} = \vec{0}$$

1)If φ is a scalar point function, $\vec{F}$ is a vector point function, then

$\nabla.(\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$

Proof:

$$\nabla \cdot (\varphi \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x}(\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \left(\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right)$$

$$= \varphi\left(\sum \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right) + \vec{F} \cdot \left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right)$$

$$\therefore \nabla \cdot (\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$$

2) If φ is a scalar point fuction, $\vec{F}$ is a vector point function, then$\nabla \times (\varphi \vec{F}) = \varphi(\nabla \times \vec{F}) + (\nabla\varphi) \times \vec{F}$

Proof:

$$\nabla \times (\varphi \vec{F}) = \sum \vec{i} \times \frac{\partial}{\partial x}(\varphi \vec{F})$$

$$= \sum \vec{i} \times \left[\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right]$$

$$= \sum \vec{i} \times \left(\frac{\partial \varphi}{\partial x}\vec{F} + \varphi\frac{\partial \vec{F}}{\partial x}\right)$$

$$= \left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right) \times \vec{F} + \varphi\left[\sum \vec{i} \times \frac{\partial \vec{F}}{\partial x}\right]$$

$$\therefore \nabla \times (\varphi \vec{F}) = \nabla\varphi \times \vec{F} + \varphi(\nabla \times \vec{F})$$

3) If $\vec{A}$ and $\vec{B}$ are vector point functions, then $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Proof:

$$\nabla.(\vec{A} \times \vec{B}) = \sum \vec{i} \cdot \frac{\partial}{\partial x}(\vec{A} \times \vec{B})$$

$$= \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \times \vec{B}\right)$$

$$= \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right) + \sum \vec{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right)$$

$$= -\left(\sum \vec{i} \times \frac{\partial \vec{B}}{\partial x}\right) \cdot \vec{A} + \left(\sum \vec{i} \times \frac{\partial \vec{A}}{\partial x}\right).\vec{B}$$

$$= -(\nabla \times \vec{B}).\vec{A} + (\nabla \times \vec{A}) \cdot \vec{B}$$

$$\therefore \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \qquad \left[\because (\nabla \times \vec{A}) \cdot \vec{B} = \vec{B} \cdot (\nabla \times \vec{A})\right]$$

(4) If $\vec{F}$ is a vector point function, then $\nabla \cdot (\nabla \times \vec{F}) = 0$.

(or)

Prove that $div(curl\,\vec{F}) = 0$.

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0$$

(5) If is a vector point function, then $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

(or)

Prove that $curl\,(curl\,\vec{F}) = grad\,(div\,\vec{F}) - \nabla^2\vec{F}$

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times (\nabla \times \vec{F}) = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

And $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

L.H.S $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z \partial x} + \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$
$$+\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

R.H.S $\nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z}\right] + \vec{j}\left[\frac{\partial^2 F_1}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y\partial z}\right] + \vec{k}\left[\frac{\partial^2 F_1}{\partial z\partial x} + \frac{\partial^2 F_2}{\partial z\partial y} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial z^2}\right] +$$

$$\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\partial z}\right]$$

L.H.S = R.H.S

$$\therefore \nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2\vec{F}$$

(6) $\nabla \cdot (\nabla\varphi) = (\nabla \cdot \nabla)\,\varphi = \nabla^2\varphi$

Proof:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\therefore \nabla \cdot (\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla\varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2\varphi$$

Example: Find (i) $\nabla \cdot \vec{r}$ (ii) $\nabla \times \vec{r}$

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

(i) $\nabla \cdot \vec{r} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}$$

Example: Find $\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right)$ where $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

Solution:

$$\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right) = \nabla \cdot \left[\frac{1}{r}\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)\right]$$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\,\vec{i} + \frac{y}{r}\,\vec{j} + \frac{z}{r}\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum \left[\frac{1}{r}(1) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum \left[\frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right)\right] \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

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$$= \sum \left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}\,(x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \qquad \because r^2 = (x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$= 2\left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right) = 2\vec{a}$$

Example: Prove that $curl(f(r)\vec{r}) = \vec{0}$

Solution:

Let $f(r)\vec{r} = f(r)\left[x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right]$

$$= xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{i}\left[zf'(r)\,\frac{\partial r}{\partial y} - yf'(r)\,\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$$

$$= \sum \vec{i}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: Prove that $curl[\varphi\,\nabla\varphi] = \vec{0}$

(or)

Prove that $\nabla \times [\varphi \nabla\varphi] = \vec{0}$

Solution:

$$\varphi\nabla\varphi = \varphi\left[\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{i}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{j}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum \vec{i}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: If $\vec{\omega}$ is a constant vector and $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$.

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$\vec{\omega} = \omega_1\,\vec{i} + \omega_2\,\vec{j} + \omega_3\,\vec{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(\omega_2 z - \omega_3 y) - \vec{j}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{i}(\omega_1 + \omega_1) - \vec{j}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1\vec{i} + 2\omega_2\vec{j} + 2\omega_3\vec{k}$$

$$= 2\left(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}\right) = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$