Lattice:

A Lattice is a partially ordered set(Poset) $(L, \leq)$ in which for every pair of elements $a, b \in L$, both greatest lower bound (GLB) and least upper bound (LUB) exists.

Note:

(i) GLB $\{a, b\} = a * b\ (or)\ a \wedge b\ (or)\ a \cdot b$

(ii) LUB $\{a, b\} = a \oplus b\ (or) a \vee b\ (or) a + b$

Properties of lattice:

Some important laws and its proof:

(i) Idempotent law:

$a \vee a = a, a \wedge a = a$

(ii) Commutative law:

$a \vee b = b \vee a\ \ and\ \ a \wedge b = b \wedge a$

(iii) Associative law:

$a \vee (b \vee c) = (a \vee b) \vee c\ \ and\ a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(iv) Absorption law:

$a \vee (a \wedge b) = a\ and\ a \wedge (a \vee b) = a$

(v) Distributive law:

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Note:

i) $a \leq a \vee b \; and \; b \leq a \vee b$

$a \vee b$ is the upper bound of a and b.

If $a \leq c \; and \; b \leq c$ then $a \vee b \leq c$

Hence $a \vee b$ is the lub of a and b.

(ii) $a \wedge b \leq a \; and \; a \wedge b \leq b$

$a \wedge b$ is the lower bound of a and b.

If $c \leq a \; and \; c \leq b$ then $c \leq a \wedge b$

Hence $a \wedge b$ is the glb of a and b.

Note:

If $a \leq b \; and \; a \leq c$ then $a \leq b \vee c$

If $a \leq b \; and \; a \leq c$ then $a \leq b \wedge c$

Problems:

1. State and prove Idempotent law:

Let $(L, \wedge, \vee)$ be given lattice. Then, for any $a, b, c \in L$,

$\boldsymbol{a \vee a = a, a \wedge a = a.}$

Proof:

Given $a \vee a =$ LUB (a, a) = LUB $(a) = a$

Hence $a \vee a = a$

Now, $a \wedge a =$ GLB (a, a) = GLB $(a) = a$

Hence $a \wedge a = a$

Hence the proof.

2. State and prove Commutative law:

Let $(L, \wedge, \vee)$ be given lattice. Then, for any $a, b, c \in L$,

$\boldsymbol{a \vee b = b \vee a \ \ and \ \ a \wedge b = b \wedge a}$

Proof:

Given $a \vee b =$ LUB (a, b) = LUB $(b, a) = b \vee a$

Hence $a \vee b = b \vee a$

Now, $a \wedge b =$ GLB (a, b) = GLB $(b, a) = b \wedge a$

Hence $a \wedge b = b \wedge a$

Hence the proof.

3. State and prove Absorption law.

(or)

Prove that $\boldsymbol{a \vee (a \wedge b) = a \text{ and } a \wedge (a \vee b) = a}$

Proof:

We have $a \wedge b \leq a \text{ and } a \leq a$

⇒ a is the upper bound of $a \wedge b$ and a.

$\Rightarrow a \vee (a \wedge b) \leq a \ldots (1)$

From the definition of lub we have

$\Rightarrow a \leq a \vee (a \wedge b) \ldots (2)$

From (1) and (2) we have $a \vee (a \wedge b) = a$

Similarly we can prove that $a \wedge (a \vee b) = a$

Hence the proof.

4. Every finite Lattice is bounded.

Proof:

Let $(L, \wedge, \vee)$ be a given lattice.

Since L is a Lattice both GLB and LUB exist.

Let "a" be GLB of L and "b" be LUB of L.

Then for any $x \in L$, we have $a \leq x \leq b$ . . . (1)

From (1)

GLB $\{a, x\}$ = $a \wedge x = a$

LUB $\{a, x\}$ = $a \vee x = x$

And

GLB $\{x, b\}$ = $x \wedge b = x$

LUB $\{x, b\}$ = $x \vee b = b$

Therefore any finite lattice is bounded.

Hence the proof.

5. State and prove Isotonicity property.

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$ then $b \leq c = \begin{cases} a \wedge b \leq a \wedge c \\ a \vee b \leq a \vee c \end{cases}$

Proof:

By consistency law we have, $a \leq b \Leftrightarrow a \wedge b = a \; and \; a \vee b = a$

Let $b \leq c \Rightarrow b \wedge c = b$ and $b \vee c = c$

Consider $(a \wedge b) \wedge (a \wedge c) = a \wedge [(b \wedge a) \wedge c]$ by Associative law

$= a \wedge [(a \wedge b) \wedge c]$ by Commutative law

$= (a \wedge a) \wedge (b \wedge c)$ by Associative law

$= a \wedge (b \wedge c)$ by Idempotent law

$= a \wedge b$ by $[b \wedge c = b]$

Hence $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

$\Rightarrow a \wedge b \leq a \wedge c$ . . . (1)

Consider $(a \vee b) \wedge (a \vee c) = a \vee [(b \vee a) \vee c]$ by Associative law

$= a \vee [(a \vee b) \vee c]$ by Commutative law

$= (a \vee a) \vee (b \vee c)$ by Associative law

$= a \vee (b \vee c)$ by Idempotent law

$= a \vee b$ by $[b \vee c = b]$

Hence $(a \vee b) \wedge (a \vee c) = a \vee b$

$\Rightarrow a \vee b \leq a \vee c$ . . . (2)

Hence the proof.

6. State and prove Distributive law.

$\boldsymbol{a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)}$

$\boldsymbol{a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)}$

Proof:

We know that $a \wedge b \leq a$ and $a \wedge b \leq b$

Also $b \leq b \vee c$

Hence $a \wedge b \leq a$ and $a \wedge b \leq b \leq b \vee c$

Hence $a \wedge b$ is the lower bound of a and $b \vee c$.

$\Rightarrow a \wedge b \leq a \wedge (b \vee c)$ ... (1)

Again $a \wedge c \leq a$ and $a \wedge c \leq c$

Also $c \leq b \vee c$

Hence $a \wedge c \leq a$ and $a \wedge c \leq c \leq b \vee c$

Hence $a \wedge c$ is the lower bound of a and $b \vee c$.

$\Rightarrow a \wedge c \leq a \wedge (b \vee c)$ ... (2)

From (1) and (2) we have

$a \wedge (b \vee c)$ is the upper bound of $a \wedge b$ and $a \wedge c$

Hence $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

$\Rightarrow a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c) \quad \ldots$ (I)

We know that $a \leq a \vee b$ and $a \leq a \vee b$

Also $b \wedge c \leq b$

Hence $a \leq a \vee b$ and $b \wedge c \leq b \leq a \vee b$

Hence $a \vee b$ is the lower bound of a and $b \wedge c$.

$\Rightarrow \ a \vee (b \wedge c) \leq a \vee b \ \ \ldots$ (3)

Again $a \leq a \vee a$ and $c \leq a \vee c$

Also $b \wedge c \leq c$

Hence $a \leq a \vee c$ and $b \wedge c \leq c \leq a \vee c$

Hence $a \vee c$ is the upper bound of a and $b \wedge c$.

$\Rightarrow a \vee (b \wedge c) \leq a \vee c \ \ \ldots$ (4)

From (3) and (4) we have

$a \vee (b \wedge c)$ is the lower bound of $a \vee b$ and $a \vee c$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \quad \ldots$ (II)

Hence the proof.

7. State and prove Cancellation law.

Let $(L, \leq)$ be a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow$

$b = c \; \forall \; a, b, c \in L$

Proof:

By absorption law, we have $a \vee (a \wedge b) = a$

Consider $b = b \vee (a \wedge b)$

$= b \vee (a \wedge c)$

$= (a \vee b) \wedge (b \vee c)$

$= (a \vee c) \wedge (b \vee c)$

$= (a \wedge b) \vee c$

$= (a \wedge c) \vee c$

$= c$

Hence the proof.

8. State and prove Consistency Law.

Let $(L, \leq)$ be a lattice. Then $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b \; \forall \; a, b, c \in L$

Proof:

First we prove that $a \leq b \Leftrightarrow a \wedge b = a$

We assume that $a \leq b$

To prove $a \wedge b = a$

We have $a \leq b$ and $a \leq a$

⇒ a is the lower bound of a and b.

$\Rightarrow a \leq a \wedge b$ . . . (1)

By the definition of greatest lower bound

$\Rightarrow a \wedge b \leq a$ . . . (2)

From (1) and (2) we have, $a = a \wedge b$

Conversely assume that $a = a \wedge b$

To prove $a \leq b$

This is possible only when $a \leq b$

Hence $a \leq b \Leftrightarrow a \wedge b = a$

Next we prove that $a \wedge b = a \Leftrightarrow a \vee b = b$

Assume that $a \wedge b = a$

To prove $a \vee b = b$

By absorption law $a \vee (a \wedge b) = a \, and \, a \wedge (a \vee b) = a$

Consider $b = b \vee (a \wedge b)$

$= b \vee a$

Hence $a \vee b = b$

Conversely assume that $a \vee b = b$

To prove $a \wedge b = a$

By absorption law $a \wedge (a \vee b) = a$

Consider $a = a \wedge (a \vee b)$

$= a \wedge b$

Hence $a \wedge b = a \Leftrightarrow a \vee b = b$

9. Show that a chain is a lattice.

Proof:

Let $(L, \leq)$ be a lattice.

If $a, b \in L$ then $a \leq b$ or $b \leq a$

If $a \leq b$ then $a \wedge b = a$ and $a \vee b = b$

Therefore GLB and LUB of a and b exists.

If $b \leq a$ then $b \wedge a = b$ and $b \vee a = a$

Therefore GLB and LUB of a and b exists.

Hence every pair of elements has a GLB and LUB.

Hence chain is lattice.