

DESIGN OF FIR FILTERS USING WINDOWING TECHNIQUES

Design of FIR Filter:

FIR Filter can be designed using three following techniques.

1. Fourier series method
2. Windowing technique
3. Frequency sampling method.

Filter design using windowing technique:

Explain the designing of FIR filters using windows. [April/May-2011]

The desired frequency response of any digital filter is periodic in frequency and can be expanded in a Fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad \text{--->(1)}$$

Where,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad \text{--->(2)}$$

Gibb's Phenomenon:

One possible way of finding an FIR filter that approximates $H(e^{j\omega})$ would be truncate the infinite Fourier series at $n = \pm \left(\frac{M-1}{2} \right)$. Abrupt truncation of the series will lead to oscillation both pass band and stop band.

This phenomenon is known as Gibbs phenomenon.

Types of window:

- Rectangular window.
- Hanning window.
- Hamming window.

Rectangular window:

The rectangular window sequence is given by,

$$w_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

Hanning window:

The hanning window sequence can be obtained by

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos \frac{2\pi n}{M-1} & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

Hamming window:

The hamming window can be obtained by

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46 \cos \frac{2\pi n}{M-1} & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

Filter coefficient ($h_d(n)$) for different types of Filters:

Type of Filter	$h_d(n)$
LPF	$h_d(n) = \frac{\omega_c}{\pi} \text{ for } n = \alpha$ $h_d(n) = \frac{\sin \omega_c (n - \alpha)}{\pi(n - \alpha)} \text{ for } n \neq \alpha$
HPF	$h_d(n) = 1 - \frac{\omega_c}{\pi} \text{ for } n = \alpha$ $h_d(n) = \frac{1}{\pi(n - \alpha)} [\sin(n - \alpha)\pi - \sin(n - \alpha)\omega_c] \text{ for } n \neq \alpha$
BPF	$h_d(n) = \frac{\omega_{c2} - \omega_{c1}}{\pi} \text{ for } n = \alpha$ $h_d(n) = \frac{1}{\pi(n - \alpha)} [\sin \omega_{c2}(n - \alpha) - \sin \omega_{c1}(n - \alpha)] \text{ for } n \neq \alpha$
BSF	$h_d(n) = 1 - \left[\frac{\omega_{c2} - \omega_{c1}}{\pi} \right] \text{ for } n = \alpha$ $h_d(n) = \frac{1}{\pi(n - \alpha)} [\sin \omega_{c1}(n - \alpha) - \sin \omega_{c2}(n - \alpha) + \sin(n - \alpha)\pi] \text{ for } n \neq \alpha$

Design an ideal low pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \leq |\omega| \leq \pi \end{cases}$$

Find the values of $h(n)$ for $M=11$ using hanning window. Find $H(z)$. Plot the magnitude and frequency response. (May/June-14)(Nov/Dec-14) (April/May 2011)(April/May-08) (Nov/Dec-09) (Nov/Dec-10)

Step 1: To find filter coefficient.

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot e^{j\omega n} d\omega \\ &= \frac{\sin(n - \alpha)\omega_c}{\pi(n - \alpha)}; \quad \because \alpha = 0 \\ h_d(n) &= \frac{\sin \frac{\pi}{2} n}{\pi n}; \quad -5 \leq n \leq 5 \end{aligned}$$

$$\text{For } n=0; \quad h_d(0) = \frac{\sin \frac{\pi}{2}(0)}{\pi(0)}$$

$$= \frac{1}{2} \frac{\sin \frac{\pi n}{2}}{\frac{\pi n}{2}}$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$h_d(0) = \frac{1}{2}$$

$$\text{For } n=1; \quad h_d(1) = h_d(-1) = \frac{\sin \frac{\pi}{2}(1)}{\pi(1)} = \frac{1}{\pi} = 0.3183$$

$$\text{For } n=2; \quad h_d(2) = h_d(-2) = \frac{\sin \pi}{2\pi} = 0$$

$$\text{For } n=3; \quad h_d(3) = h_d(-3) = \frac{\sin \frac{3\pi}{2}}{3\pi} = -0.106$$

$$\text{For } n=4; \quad h_d(4) = h_d(-4) = \frac{\sin 2\pi}{4\pi} = 0$$

$$\text{For } n=5; \quad h_d(5) = h_d(-5) = \frac{\sin \frac{5\pi}{2}}{5\pi} = 0.06366$$

Step 2: To find hanning window:

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos \frac{2\pi n}{M-1} & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

$$= 0.5 + 0.5 \cos \frac{2\pi n}{10} \quad \text{for } -5 \leq n \leq (5)$$

$$\text{For } n=0; \quad w_{Hn}(0) = 0.5 + 0.5 \cos \frac{\pi(0)}{5} = 1$$

$$\text{For } n=1; \quad w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5 \cos \frac{\pi(1)}{5} = 0.9045$$

$$\text{For } n=2; \quad w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5 \cos \frac{\pi(2)}{5} = 0.6545$$

$$\text{For } n=3; \quad w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5 \cos \frac{\pi(3)}{5} = 0.3454$$

$$\text{For } n=4; \quad w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5 \cos \frac{\pi(4)}{5} = 0.0954$$

$$\text{For } n=5; \quad w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5 \cos \frac{\pi(5)}{5} = 0$$

Step 3: To find filter coefficients using hanning window are

$$h(n) = h_d(n) * w_{Hn}(n) \quad \text{for } -5 \leq n \leq 5$$

$$h(0) = h_d(0) * w_{Hn}(0) = (0.5) * 1 = 0.5$$

$$h(1) = h_d(1) * w_{Hn}(1) = 0.3183 * 0.9045 = 0.2879$$

$$h(2) = h_d(2) * w_{Hn}(2) = 0 * 0.6545 = 0$$

$$h(3) = h_d(3) * w_{Hn}(3) = -0.106 * 0.3454 = 0.0366$$

$$h(4) = h_d(4) * w_{Hn}(4) = 0.0636 * 0.0954 = 0.00606$$

$$h(5) = h_d(5) * w_{Hn}(5) = 0 * 0.65 = 0$$

Step 4: The transfer function of the filter is given by

$$\begin{aligned}
 H(z) &= h(0) + \sum_{n=1}^{\frac{M-1}{2}} [h(n)(z^n + z^{-n})] \\
 &= 0.5 + \sum_{n=1}^5 h(n)(z^n + z^{-n}) \\
 &= 0.5 + h(1)(z^1 + z^{-1}) + h(2)(z^2 + z^{-2}) + h(3)(z^3 + z^{-3}) + h(4)(z^4 + z^{-4}) + h(5)(z^5 + z^{-5}) \\
 &= 0.5 + 0.287z^1 + 0.287z^{-1} + 0.0366z^3 + 0.0366z^{-3} + 0.006z^4 + 0.006z^{-4}
 \end{aligned}$$

Step 5 : The transfer function of the realizable filter is

$$\begin{aligned}
 H'(z) &= z^{-\frac{(M-1)}{2}} H(z) \\
 &= z^{-5} [0.5 + 0.287z^1 + 0.287z^{-1} + 0.0366z^3 + 0.0366z^{-3} + 0.006z^4 + 0.006z^{-4}] \\
 H'(z) &= 0.5z^{-5} + 0.287z^{-4} + 0.287z^{-6} + 0.0366z^{-2} + 0.0366z^{-8} + 0.006z^{-1} + 0.006z^{-9}
 \end{aligned}$$

The filter coefficients of causal filter are given by

$$h(0) = h(3) = h(7) = h(10) = 0, h(1) = h(9) = 0.006; h(2) = h(8) = 0.0366; h(4) = h(6) = 0.287; h(5) = 0.5$$

Step 6 : The frequency response is given by

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^5 a(n) \cos \omega n$$

where,

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = 2(0.006) = 0.12$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = 2 * 0.036 = 0.072$$

$$a(4) = 2h(5-4) = 2h(1) = 2 * 0.006 = 0.12$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^5 a(n) \cos \omega n$$

$$= a(0) + a(1)\cos \omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega$$

$$= 0.5 + 0.12 \cos \omega + 0.072 \cos 3\omega + 0.12 \cos 4\omega$$

Magnitude in dB is calculated by varying 0 to 10 and tabulated below.

ω (in degree)	0	1	2	3	4	5
$\bar{H}(e^{j\omega})$	0.812	0.8115	0.810	0.8083	0.8054	0.8018
$ H(e^{j\omega}) _{dB}$	-1.8	-1.814	-1.83	-1.85	-1.88	-1.91

Design an ideal high pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{4} \leq |\omega| \leq \pi \\ 0 & \text{for } |\omega| \leq \frac{\pi}{4} \end{cases}$$

Find the values of $h(n)$ for $N=11$ using hanning window. (May/June-16)(April/May-08)

Solution:

Given :

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } \frac{\pi}{4} \leq |\omega| \leq \pi \\ 0 & \text{for } |\omega| \leq \frac{\pi}{4} \end{cases} \quad \text{Hence } \omega_c = \frac{\pi}{4}$$

Step 1: To find filter coefficient.

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot e^{j\omega n} d\omega \end{aligned}$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)} [\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c] \text{ for } n \neq \alpha$$

$$h_d(n) = \frac{1}{\pi n} \left[\sin n\pi - \sin \frac{n\pi}{4} \right]; \quad -5 \leq n \leq 5$$

Step 2 :

$$\text{For } n=0 \quad h_d(0) = \frac{1}{\pi(0)} \left[\sin(0)\pi - \sin \frac{(0)\pi}{4} \right]$$

$$\text{Using, L' hospital rule} \quad h_d(0) = 1 - \frac{1}{4} = 0.75$$

$$\text{For } n=1; \quad h_d(1) = h_d(-1) = \frac{1}{\pi(1)} \left[\sin(1)\pi - \sin \frac{(1)\pi}{4} \right] = -0.225$$

$$\text{For } n=2; \quad h_d(2) = h_d(-2) = \frac{1}{\pi(2)} \left[\sin(2)\pi - \sin \frac{(2)\pi}{4} \right] = -0.159$$

$$\text{For } n=3; \quad h_d(3) = h_d(-3) = \frac{1}{\pi(3)} \left[\sin(3)\pi - \sin \frac{(3)\pi}{4} \right] = -0.075$$

$$\text{For } n=4; \quad h_d(4) = h_d(-4) = \frac{1}{\pi(4)} \left[\sin(4)\pi - \sin \frac{(4)\pi}{4} \right] = 0$$

$$\text{For } n=5; \quad h_d(5) = h_d(-5) = \frac{1}{\pi(5)} \left[\sin(5)\pi - \sin \frac{(5)\pi}{4} \right] = 0.045$$

Step 2: Using Hanning window:

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5 \cos \frac{2\pi n}{M-1}, & \left(\frac{M-1}{2} \right) \leq n \leq \left(\frac{M-1}{2} \right) \\ 0 & \text{otherwise} \end{cases}$$

$$w_{Hn}(0) = 0.5 + 0.5 = 1$$

$$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5 \cos \left(\frac{\pi(1)}{5} \right) = 0.9045$$

$$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5 \cos \left(\frac{\pi(2)}{5} \right) = 0.655$$

$$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5 \cos \left(\frac{\pi(3)}{5} \right) = 0.345$$

$$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5 \cos \left(\frac{\pi(4)}{5} \right) = 0.0945$$

$$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5 \cos \left(\frac{\pi(5)}{5} \right) = 0$$

Step 3: The filter coefficients using hanning window are,

$$h(n) = h_d(n)w_{Hn}(n) \quad \text{for } -5 \leq n \leq 5$$

$$h(0) = h_d(0)w_{Hn}(0) = (0.75)(0) = 0$$

$$h(1) = h_d(1)w_{Hn}(1) = (-0.225)(0.905) = -0.204$$

$$h(2) = h_d(2)w_{Hn}(2) = (-0.159)(0.655) = -0.104$$

$$h(3) = h_d(3)w_{Hn}(3) = (-0.075)(0.345) = -0.026$$

$$h(4) = h_d(4)w_{Hn}(4) = (0)(0.8145) = 0$$

$$h(5) = h_d(5)w_{Hn}(5) = (0.045)(0) = 0$$

Step 4: The transfer function of the filter is given by

The transfer function of the realizable filter is

$$\begin{aligned} H'(z) &= z^{-\left(\frac{M-1}{2}\right)} H(z) \\ &= z^{-5} [0.75 - 0.204z^{-1} - 0.204z^{-1} - 0.104z^{-2} - 0.104z^{-2} - 0.026z^{-3} - 0.026z^{-3}] \\ &= 0.75z^{-5} - 0.204z^{-6} - 0.204z^{-4} - 0.104z^{-7} - 0.104z^{-3} - 0.026z^{-8} - 0.026z^{-2} \end{aligned}$$

$$\begin{aligned} H(z) &= h(0) + \sum_{n=1}^{M-1} h(n)(z^{-n} + z^n) \\ &= h(0) + \sum_{n=1}^5 h(n)[z^{-n} + z^n] \\ &= 0.75 + h(1)[z^{-1} + z^1] + h(2)[z^{-2} + z^2] + h(3)[z^{-3} + z^3] + h(4)[z^{-4} + z^4] + h(5)[z^{-5} + z^5] \\ &= 0.75 - 0.204z^{-1} - 0.204z^1 - 0.104z^{-2} - 0.104z^2 - 0.026z^{-3} - 0.026z^3 \end{aligned}$$

The causal filter coefficients are

$$h(0) = h(1) = h(9) = h(10) = 0;$$

$$h(2) = h(8) = -0.026$$

$$h(3) = h(7) = -0.104$$

$$h(4) = h(6) = -0.204$$

$$h(5) = 0.75$$

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} a(n) \cos \omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{M-1}{2} - n\right] = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = -0.408$$

$$a(2) = 2h(5-2) = 2h(3) = -0.208$$

$$a(3) = 2h(5-3) = 2h(2) = -0.052$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.408 \cos \omega - 0.208 \cos 2\omega - 0.052 \cos 3\omega$$

ω (in degrees)	0	1	2	3	4	5
$\bar{H}(e^{j\omega})$	0.082	0.0822	0.083	0.08433	0.08615	0.08848
$ H(e^{j\omega}) _{dB}$	-21.72	-21.70	-21.61	-21.480	-21.29	-21.11

b) Using Hamming window:

The hamming window sequence is given by

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi n}{M-1}\right); & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{\pi n}{5}\right); & \text{for } -5 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$$w_{Hm}(0) = 1$$

$$w_{Hm}(1) = w_{Hm}(-1) = 0.912$$

$$w_{Hm}(2) = w_{Hm}(-2) = 0.682$$

$$w_{Hm}(3) = w_{Hm}(-3) = 0.398$$

$$w_{Hm}(4) = w_{Hm}(-4) = 0.1678$$

$$w_{Hm}(5) = w_{Hm}(-5) = 0.08$$

The coefficients using hamming window sequence are,

$$h(n) = h_d(n) * w_{Hm}(n); \quad -5 \leq n \leq 5$$

$$h(0) = h_d(0) * w_{Hm}(0) = 1 * 0.75 = 0.75$$

$$h(1) = h_d(1) * w_{Hm}(1) = (-0.225) * (0.912) = -0.2052$$

$$h(2) = h_d(2) * w_{Hm}(2) = (-0.159) * (0.682) = -0.1084$$

$$h(3) = h_d(3) * w_{Hm}(3) = (-0.075) * (0.398) = -0.03$$

$$h(4) = h_d(4) * w_{Hm}(4) = (0) * (0.16787) = 0$$

$$h(5) = h_d(5) * w_{Hm}(5) = (-0.045) * (0.08) = 0.0036$$

The transfer function of the filter is given by

$$\begin{aligned} H(z) &= h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n) [z^{-n} + z^n] \\ &= h(0) + \sum_{n=1}^5 h(n) [z^{-n} + z^n] \\ &= 0.75 + h(1)[z^{-1} + z^1] + h(2)[z^{-2} + z^2] + h(3)[z^{-3} + z^3] + h(4)[z^{-4} + z^4] + h(5)[z^{-5} + z^5] \\ &= 0.75 - 0.2052z^{-1} - 0.2052z^1 - 0.1084z^{-2} - 0.1084z^2 - 0.03z^{-3} - 0.03z^3 + 0.0036z^{-5} + 0.0036z^5 \end{aligned}$$

The transfer function of the realizable filter is

$$\begin{aligned} H'(z) &= z^{-\left(\frac{M-1}{2}\right)} H(z) \\ &= z^{-5} [0.75 - 0.2052z^{-1} - 0.2052z^1 - 0.1084z^{-2} - 0.1084z^2 - 0.03z^{-3} - 0.03z^3 + 0.0036z^{-5} + 0.0036z^5] \\ H'(z) &= 0.75z^{-5} - 0.2052z^{-6} - 0.2052z^{-4} - 0.1084z^{-7} - 0.1084z^{-3} - 0.03z^{-8} - 0.03z^{-2} + 0.0036z^{-10} + 0.0036z^0 \end{aligned}$$

The filter coefficients of causal filter are

$$h(0) = h(10) = 0.0036; h(1) = h(9) = 0; h(2) = h(8) = -0.03; h(3) = h(7) = -0.1084; h(4) = h(6) = -0.2052; h(5) = 0.75$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n) \cos \omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = -0.4104$$

$$a(2) = 2h(5-2) = 2h(3) = -0.2168$$

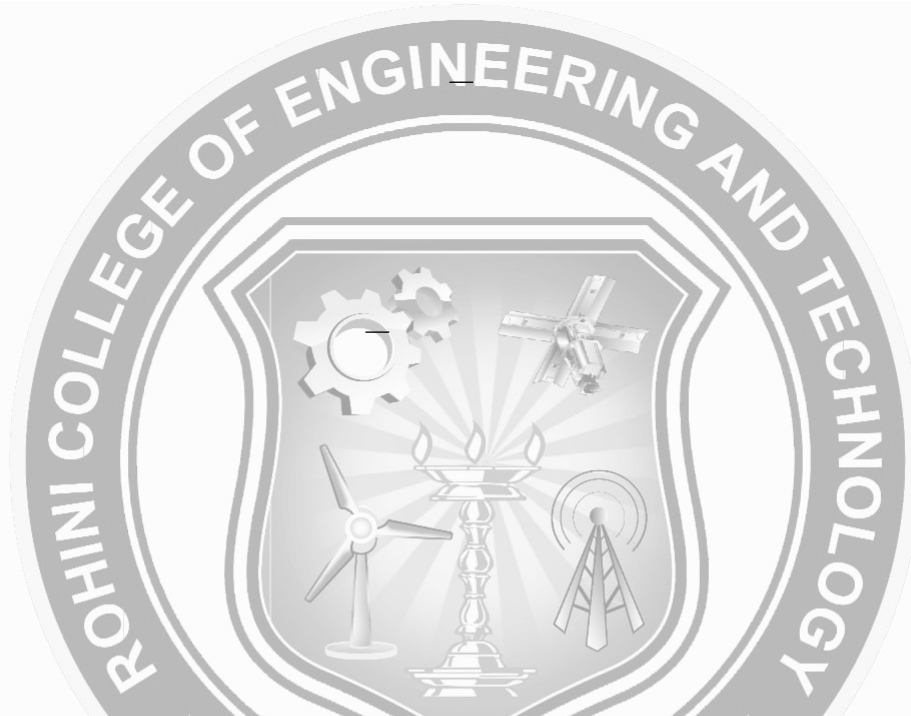
$$a(3) = 2h(5 - 3) = 2h(2) = -0.06$$

$$a(4) = 2h(5 - 4) = 2h(1) = 0$$

$$a(5) = 2h(5 - 5) = 2h(0) = 0.0072$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.4104 \cos \omega - 0.2168 \cos 2\omega - 0.06 \cos 3\omega + 0.0072 \cos 5\omega$$

ω (in degrees)	0	30	60	90	120	150	180
$\bar{H}(e^{j\omega})$	0.07	0.28	0.7168	0.9668	1	1.003	1.0108
$ H(e^{j\omega}) _{dB}$	-23.1	-11	-2.89	-0.29	0	0.028	0.093



For a FIR linear phase digital filter approximating the ideal frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \leq |\omega| \leq \pi \end{cases}$$

Determine the coefficients of a 5 tap filter using rectangular window.

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \leq |\omega| \leq \pi \end{cases}$$

$$\text{Hence } \omega_c = \frac{\pi}{6}; N = 5.$$

Step 1: To find filter coefficient.

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1 \cdot e^{j\omega n} d\omega \\ &= \frac{\sin(n - \alpha)\omega_c}{\pi(n - \alpha)}; \end{aligned}$$

$$\because \alpha = 0$$

$$h_d(n) = \frac{\sin \frac{\pi}{6} n}{\pi n}; \quad -2 \leq n \leq 2$$

For $n=0$:
$$h_d(0) = \frac{\sin \frac{\pi}{6} (0)}{\pi(0)}$$

$$= \frac{1 \sin \frac{\pi n}{6}}{6 \frac{\pi n}{6}}$$

$$h_d(0) = \frac{1}{6} = 0.16 \quad \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

for $n=1$
$$h_d(1) = h_d(-1) = \frac{\sin \frac{\pi}{6} (1)}{\pi(1)} = \frac{0.5}{\pi} = 0.159$$

For $n=2$;
$$h_d(2) = h_d(-2) = \frac{\sin \frac{\pi}{6} (2)}{2\pi} = 0.1379$$

Step 2: Using Rectangular window:

$$w_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

$$= 1 \quad \text{for } -2 \leq n \leq 2$$

$$w_R(0) = w_R(1) = w_R(2) = 1$$

Step 3: To find filter coefficients using rectangular window are

$$h(n) = h_d(n) * w_{Hn}(n) \quad \text{for } -2 \leq n \leq 2$$

$$h(0) = h_d(0) * w_{Hn}(0) = (0.16) * 1 = 0.16$$

$$h(1) = h_d(1) * w_{Hn}(1) = 0.59 * 1 = 0.59$$

$$h(2) = h_d(2) * w_{Hn}(2) = 0.137 * 1 = 0.137$$

Step 4: The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{M-1} [h(n)(z^n + z^{-n})]$$

$$= 0.5 + \sum_{n=1}^2 h(n)(z^n + z^{-n})$$

$$= 0.16 + h(1)(z^1 + z^{-1}) + h(2)(z^2 + z^{-2})$$

$$= 0.16 + 0.59z^1 + 0.59z^{-1} + 0.137z^2 + 0.137z^{-2}$$

Step 5: The transfer function of the realizable filter is

$$H'(z) = z^{-\frac{(M-1)}{2}} H(z)$$

$$= z^{-2} [0.16 + 0.59z^1 + 0.59z^{-1} + 0.137z^2 + 0.137z^{-2}]$$

$$H'(z) = 0.16z^{-2} + 0.59z^{-1} + 0.59z^{-3} + 0.137z^{-4} + 0.137$$

The filter coefficients of causal filter are given by

$$h(0) = h(4) = 0.137, h(1) = h(3) = 0.59; h(2) = 0.16; h(5) = 0$$

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} a(n) \cos \omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(2) = 0.32$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(2-n)$$

$$a(1) = 2h(2-1) = 2h(1) = 1.18$$

$$a(2) = 2h(2-2) = 2h(0) = 0.274$$

$$H(e^{j\omega}) = 0.32 + 1.18\cos\omega + 0.274\cos 2\omega$$

ω (in degrees)	0	30	60	90	120	150	180
$\bar{H}(e^{j\omega})$	1.774	1.47	0.773	0.046	-0.407	-0.564	-0.586

Design an ideal band pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

Find the values of $h(n)$ for $N=11$ using rectangular window.

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_{c1} = \frac{\pi}{4} \text{ and } \omega_{c2} = \frac{3\pi}{4}$$

Step 1: Filter coefficients are,

$$h_d(n) = \frac{1}{\pi(n-\alpha)} [\sin \omega_{c2}(n-\alpha) - \sin \omega_{c1}(n-\alpha)] \text{ for } n \neq \alpha$$

$$= \frac{1}{\pi n} \left[\sin \frac{3\pi n}{4} - \sin \frac{\pi n}{4} \right]$$

For $n = 0$;

$$h_d(0) = 0.5$$

$$h_d(1) = h_d(-1) = \frac{\sin \frac{3\pi(1)}{4} - \sin \frac{\pi(1)}{4}}{\pi(1)} = 0$$

$$h_d(2) = h_d(-2) = \frac{\sin \frac{3\pi(2)}{4} - \sin \frac{\pi(2)}{4}}{\pi(2)} = -0.3183$$

$$h_d(3) = h_d(-3) = \frac{\sin \frac{3\pi(3)}{4} - \sin \frac{\pi(3)}{4}}{\pi(3)} = 0$$

$$h_d(4) = h_d(-4) = \frac{\sin \frac{3\pi(4)}{4} - \sin \frac{\pi(4)}{4}}{\pi(4)} = 0$$

$$h_d(5) = h_d(-5) = \frac{\sin \frac{3\pi(5)}{4} - \sin \frac{\pi(5)}{4}}{\pi(5)} = 0$$

Step 2: Using rectangular window

$$w_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

$$= 1 \quad \text{for } -2 \leq n \leq 2$$

$$w_R(0) = w_R(1) = w_R(2) = w_R(3) = w_R(4) = w_R(5) = 1$$

Step 3: Filter coefficients using rectangular window

$$h(n) = w_R(n) * h_d(n); \quad -5 \leq n \leq 5$$

$$h(0) = w_R(0) * h_d(0) = 1 * 0.5 = 0.5$$

$$h(1) = w_R(1) * h_d(1) = 1 * 0 = 0$$

$$h(2) = w_R(2) * h_d(2) = 1 * -0.3183 = -0.3183$$

$$h(3) = w_R(3) * h_d(3) = 1 * 0 = 0$$

$$h(4) = w_R(4) * h_d(4) = 1 * 0 = 0$$

$$h(5) = w_R(5) * h_d(5) = 1 * 0 = 0$$

Step 4: The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)[z^{-n} + z^n]$$

$$= 0.5 - 0.3183z^2 - 0.3183z^{-2}$$

Step 5: The transfer function of the realizable filter is

$$H'(z) = z^{-5}(0.5 - 0.3183z^2 - 0.3183z^{-2})$$

$$= 0.5z^{-5} - 0.3183z^{-3} - 0.3183z^{-7}$$

The filter coefficients of the causal filters are

$$h(0) = h(10) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$$

$$h(3) = h(7) = -0.3183$$

$$h(5) = 0.5$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n) \cos \omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = 0$$

$$a(2) = 2h(5-2) = 2h(3) = -0.6366$$

$$a(3) = 2h(5-3) = 2h(2) = 0$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = 0.5 - 0.6366 \cos 2\omega$$

ω (in degrees)	0	30	60	90	120	150	180
$\bar{H}(e^{j\omega})$	-0.1366	0.1817	0.818	1.1366	0.818	0.1817	-0.1366
$ H(e^{j\omega}) _{dB}$	-17.3	-14.8	-1.74	1.11	-1.74	-14.8	-17.3

Design an ideal band Reject filter with a frequency response Find the values of $h(n)$ for $N=11$ using rectangular window.

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{\pi}{3} \text{ and } |\omega| \leq \frac{2\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{\pi}{3} \text{ and } |\omega| \leq \frac{2\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

$$\omega_{c1} = \frac{\pi}{3} \text{ and } \omega_{c2} = \frac{2\pi}{3}$$

Step 1: Filter coefficients are,

$$h_d(n) = \frac{1}{\pi(n-\alpha)} \left[\sin \omega_{c1}(n-\alpha) - \sin \omega_{c2}(n-\alpha) + \sin(n-\alpha)\pi \right]$$

$$= \frac{1}{\pi n} \left[\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3} + \sin n\pi \right]$$

For $n = 0$;

$$h_d(0) = 0.667$$

$$h_d(1) = h_d(-1) = \frac{\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3} + \sin n\pi}{\pi(1)} = 0$$

$$h_d(2) = h_d(-2) = \frac{\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3} + \sin n\pi}{\pi(2)} = 0.2757$$

$$h_d(3) = h_d(-3) = \frac{\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3} + \sin n\pi}{\pi(3)} = 0$$

$$h_d(4) = h_d(-4) = \frac{\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3} + \sin n\pi}{\pi(4)} = -0.1378$$

$$h_d(5) = h_d(-5) = \frac{\sin \frac{\pi n}{3} - \sin \frac{2\pi n}{3} + \sin n\pi}{\pi(5)} = 0$$

Step 2: Using rectangular window

$$w_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{M-1}{2}\right) \leq n \leq \left(\frac{M-1}{2}\right) \\ 0 & \text{for otherwise.} \end{cases}$$

$$= 1 \quad \text{for } -2 \leq n \leq 2$$

$$w_R(0) = w_R(1) = w_R(2) = w_R(3) = w_R(4) = w_R(5) = 1$$

Step 3: Filter coefficients using rectangular window

$$h(n) = w_R(n) * h_d(n); \quad -5 \leq n \leq 5$$

$$h(0) = w_R(0) * h_d(0) = 1 * 0.667 = 0.667$$

$$h(1) = w_R(1) * h_d(1) = 1 * 0 = 0$$

$$h(2) = w_R(2) * h_d(2) = 1 * 0.2757 = 0.2757$$

$$h(3) = w_R(3) * h_d(3) = 1 * 0 = 0$$

$$h(4) = w_R(4) * h_d(4) = 1 * -0.1378 = -0.1378$$

$$h(5) = w_R(5) * h_d(5) = 1 * 0 = 0$$

Step 4: The transfer function of the filter is

$$\begin{aligned} H(z) &= h(0) + \sum_{n=1}^{M-1} h(n)[z^{-n} + z^n] \\ &= 0.667 + 0.2757z^2 + 0.2757z^{-2} - 0.1378z^4 - 0.1378z^{-4} \end{aligned}$$

Step 5: The transfer function of the realizable filter is

$$\begin{aligned} H'(z) &= z^{-5}(0.667 + 0.2757z^2 + 0.2757z^{-2} - 0.1378z^4 - 0.1378z^{-4}) \\ &= 0.667z^{-5} + 0.2757z^{-3} + 0.2757z^{-7} - 0.1378z^{-9} - 0.1378z^{-1} \end{aligned}$$

The filter coefficients of the causal filter are

$$h(0) = h(10) = h(2) = h(8) = h(4) = h(6) = 0$$

$$h(1) = h(9) = -0.1378$$

$$h(3) = h(7) = 0.2757$$

$$h(5) = 0.667$$

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} a(n) \cos \omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.667$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = 0$$

$$a(2) = 2h(5-2) = 2h(3) = 0.5514$$

$$a(3) = 2h(5-3) = 2h(2) = 0$$

$$a(4) = 2h(5-4) = 2h(1) = -0.2756$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = 0.667 + 0.5514 \cos 2\omega - 0.2756 \cos 4\omega$$

ω (in degrees)	0	30	60	90	120	150	180
$\bar{H}(e^{j\omega})$	0.9428	1.08	0.526	16	0.529	1.08	0.9428
$ H(e^{j\omega}) _{dB}$	-0.5	0.67	-5.53	-15.9	-5.53	0.67	-0.5

Design a high pass filter using window, with a cut-off frequency of 1.2 radians/sec and N=9. [Nov/Dec-2016]

SOLUTION:

Given

$$\Omega_c = 1.2 \text{ radians / sec}$$

$$\text{if } T=1 \text{ sec}$$

$$\omega_c = \Omega_c T = 1.2 \text{ radians}$$

The impulse response of a high pass filter with a cut off frequency ω_c is

$$h_d(n) = \frac{-\sin \omega_c n}{\pi} \quad |n| > 0$$

$$1 - \frac{\omega_c}{\pi} \quad \text{for } n=0$$

$$\omega_c = 1.2$$

$$h_d(0) = 1 - \frac{1.2}{\pi} = 0.618$$

$$h_d(-1) = h_d(1) = \frac{-\sin 1.2}{\pi} = -0.2966$$

$$h_d(-2) = h_d(2) = \frac{-\sin 2.4}{2\pi} = -0.1075$$

$$h_d(-3) = h_d(3) = \frac{-\sin 3.6}{3\pi} = 0.0469$$

$$h_d(-4) = h_d(4) = \frac{-\sin 4.8}{4\pi} = 0.0719$$

Hamming window for $-4 \leq n \leq 4$ is

$$w_H(n) = 0.54 + 0.46 \cos \frac{2\pi n}{8} \quad \text{for } -4 \leq n \leq 4$$

$$= 0.54 + 0.46 \cos \frac{\pi n}{4}$$

$$w_H(0) = 1$$

$$w_H(-1) = w_H(1) = 0.865$$

$$w_H(-2) = w_H(2) = 0.54$$

$$w_H(-3) = w_H(3) = 0.215$$

$$w_H(-4) = w_H(4) = 0.08$$

$$h(n) = h_d(n) w_H(n)$$

$$h(0) = (0.618)(1) = 0.618$$

$$h(-1) = h(1) = (-0.2966)(0.865) = -0.256$$

$$h(-2) = h(2) = -0.058$$

$$h(-3) = h(3) = 0.01$$

$$h(-4) = h(4) = 0.0057$$

The casual filter coefficients are

$$h(0) = (0.618)(1) = 0.618$$

$$h(-1) = h(1) = (-0.2966)(0.865) = -0.256$$

$$h(-2) = h(2) = -0.058$$

$$h(-3) = h(3) = 0.01$$

$$h(-4) = h(4) = 0.0057$$