3.5 RANDOM TELEGRAPH SIGNAL

Define semi random telegraph signal process and random telegraph signal process and Prove also that the former is evolutionary and later is WSS.

Sol :

If {N(t) } is a poisson process and X(t) = $(-1)^{N(t)}$, then {X(t)} is called a semi random telegraph signal process.

$$\therefore X(t) = \begin{cases} -1; & N(t) is\ odd \\ +1 & N(t)\ is\ even \end{cases}$$

Since N(t) is a poisson process, its probability law is

$$P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,3, \dots \dots \dots, \infty$$

$$P[X(t) = -1] = P[N(t)\ is\ odd]$$

$$= P[N(t) = 1] + P[N(T) = 3] + P[N(t) = 5] + \dots\dots$$

$$= \frac{e^{-\lambda t}(\lambda t)^1}{1!} + \frac{e^{-\lambda t}(\lambda t)^3}{3!} + \frac{e^{-\lambda t}(\lambda t)^5}{5!} + \dots\dots\dots$$

$$= e^{-\lambda t}\left[\frac{\lambda t}{1!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^5}{5!} + \cdots \dots\dots\right] \left(\text{since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \text{Sinhx} = \frac{x}{1!} + \frac{x^3}{3!} + \cdots, \right.$$

$$\left. Coshx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right)$$

$$P[X(t) = -1] = e^{-\lambda t} \sinh \lambda t$$

$$P[X(t) = 1] = P[N(t)\ is\ even]$$

$$= P[N(t) = 0] + P[N(T) = 2] + P[N(t) = 4] + \dots\dots$$

$$= e^{-\lambda t} + \frac{e^{-\lambda t}(\lambda t)^2}{2!} + \frac{e^{-\lambda t}(\lambda t)^4}{4!} + \dots\dots\dots$$

$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots \ldots..\right]$ (since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, Sinhx = $\frac{x}{1!} + \frac{x^3}{3!} + \cdots,$

$$Coshx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad)$$

P[X(t) = 1] = $e^{-\lambda t}$ Cosh λt

The mean of the process is given by

E[X(t)] = $\sum nP_n\,(t)$

= (-1)$e^{-\lambda t} sinh\lambda t + e^{-\lambda t} cosh\lambda t$

= $e^{-\lambda t}[cosh\lambda t - sinh\lambda t]$

= $e^{-\lambda t}\left[\left(\frac{e^{\lambda t}+e^{-\lambda t}}{2}\right) - \left(\frac{e^{\lambda t}-e^{-\lambda t}}{2}\right)\right]$

= = $e^{-\lambda t}[\frac{e^{\lambda t}}{2} + \frac{e^{-\lambda t}}{2} - \frac{e^{\lambda t}}{2} + \frac{e^{-\lambda t}}{2}]$

= $e^{-\lambda t}e^{-\lambda t}$

E[X(t)] = $e^{-2\lambda t}$ which is not a constant.

$\therefore \{X(t)\}$ is not stationary.

Hence semi random telegraph signal is an evolutionary process.

The autocorrelation function is given by

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$X(t)X(t+\tau) = \begin{cases} -1; & N(\tau) is\ odd \\ +1 & N(\tau)\ is\ even \end{cases}$$

P[$X(t)X(t+\tau) = -1] = P[N(\tau) is\ odd]$

$$P[X(t)X(t+\tau) = -1] = P[N(\tau) is\ odd]$$

$$= e^{-\lambda\tau} sinh\lambda\tau$$

$$P[X(t)X(t+\tau) = 1] = P[N(\tau) is\ even]$$

$$= e^{-\lambda\tau} cosh\lambda\tau$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$= (-1)e^{-\lambda\tau} sinh\lambda\tau + e^{-\lambda\tau} cosh\lambda\tau$$

$$= e^{-\lambda\tau}[cosh\lambda\tau - sinh\lambda\tau]$$

$$= e^{-\lambda\tau}\left[\left(\frac{e^{\lambda\tau}+e^{-\lambda\tau}}{2}\right) - \left(\frac{e^{\lambda\tau}-e^{-\lambda\tau}}{2}\right)\right]$$

$$= e^{-\lambda\tau}\left[\frac{e^{\lambda\tau}}{2} + \frac{e^{-\lambda\tau}}{2} - \frac{e^{\lambda\tau}}{2} + \frac{e^{-\lambda\tau}}{2}\right]$$

$$= e^{-\lambda\tau} e^{-\lambda\tau}$$

$$= e^{-2\lambda\tau}$$

RANDOM TELEGRAPH SIGNAL

Let {X(t)} be semi random telegraph signal process and Y(t) = $\alpha X(t)$ where α is a R.V which takes value -1 and +1 with probability $\frac{1}{2}$ and $\frac{1}{2}$ respectively, which is independent of { X(t)}, then {Y(t)} is called a random telegraph signal process.

Probability distribution for α

α	-1	1
P($\boldsymbol{\alpha}$)	1/2	1/2

$$E(\alpha) = \sum \alpha P(\alpha) = -\frac{1}{2} + \frac{1}{2} = 0$$

$$E(\alpha^2) = \sum \alpha^2 P(\alpha) = \frac{1}{2} + \frac{1}{2} = 1$$

To prove {X(t)} is a WSS process

$$E[Y(t)] = E[\alpha X(t)]$$

$= E(\alpha)E[X(t)]$ Since α is independent of X(t)

$= 0.\, e^{-2\lambda t}$ Since $E(\alpha) = 0$

= 0

$\therefore E[Y(t)]$ is a constant

$$R_{XY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E[\alpha X(t)\alpha X(t+\tau)]$$

$$= E[\alpha^2 X(t)X(t+\tau)]$$

$$= E(\alpha^2)E[X(t)X(t+\tau)]$$

$= 1 \times R_{XX}(\tau)$

$R_{YY}(\tau) = e^{-2\lambda\tau}$ which is a function of τ

$\therefore \{Y(t)\} is\ WSS.$

Hence the random telegraph signal process is WSS.