1.4 THEOREMS AND APPLICATIONS

DIVERGENCE THEOREM:

Divergence of a Vector

The divergence of $\boldsymbol{A}$ at a given point $\boldsymbol{P}$ is the outward flux per unit volume as the volume shrinks about $\boldsymbol{P}$**.**

The divergence of a vector $\boldsymbol{A}$ at any point is defined as the limit of its surface integrated per unit volume as the volume enclosed by the surface shrinks to zero

$$\boldsymbol{\nabla . A = \lim_{v \to 0} \frac{1}{v} \oiint_{s} A . \vec{n}\, ds}$$

It can be expressed as

$$\boldsymbol{\nabla = \frac{\partial}{\partial_x} \overrightarrow{a_x} + \frac{\partial}{\partial_y} \overrightarrow{a_y} + \frac{\partial}{\partial_z} \overrightarrow{a_z}}$$

$$\boldsymbol{A = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z}$$

$$\boldsymbol{\nabla . A = \left(\frac{\partial}{\partial_x} \overrightarrow{a_x} + \frac{\partial}{\partial_y} \overrightarrow{a_y} + \frac{\partial}{\partial_z} \overrightarrow{a_z} \right) + \left(A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z \right)}$$

$$\boldsymbol{\nabla . \mathrm{A} = \frac{\partial A_x}{\partial_x} + \frac{\partial A_y}{\partial_y} + \frac{\partial A_z}{\partial_z}}$$

This operation is called divergence

$$\boldsymbol{\nabla . A = div\, A}$$

Divergence of a vector is a scalar quantity.

DIVERGENCE THEOREM

The volume integral of the divergence of a vector field over a volume is equal to the surface integral of the normal component of this vector over the surface bounding the volume.

$$\iiint_{v} \nabla . A\, dv = \oiint_{s} A . ds$$

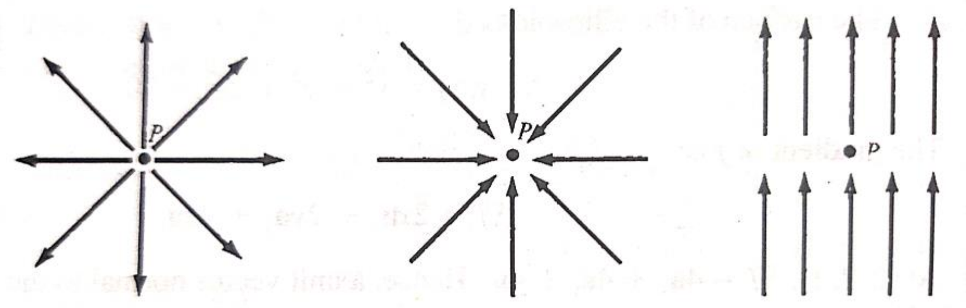

Figure 1.4.1 Illustration of the divergence of a vector at P

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-72]

Proof:

The divergence of any vector $\boldsymbol{A}$ is given by

$$\nabla = \frac{\partial}{\partial_x}\overrightarrow{a_x} + \frac{\partial}{\partial_y}\overrightarrow{a_y} + \frac{\partial}{\partial_z}\overrightarrow{a_z}$$

$$A = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

$$\nabla . A = \left(\frac{\partial}{\partial_x}\overrightarrow{a_x} + \frac{\partial}{\partial_y}\overrightarrow{a_y} + \frac{\partial}{\partial_z}\overrightarrow{a_z}\right) + \left(A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z\right)$$

$$\nabla . \mathrm{A} = \frac{\partial A_x}{\partial_x} + \frac{\partial A_y}{\partial_y} + \frac{\partial A_z}{\partial_z}$$

Divergence Theorem

$$\iiint_v \nabla . A \, dv = \oiint_s A . ds$$

Take the volume integral on both sides

$$\iiint_v \nabla . A \, dv = \iiint_v \left[\frac{\partial A_x}{\partial_x} + \frac{\partial A_y}{\partial_y} + \frac{\partial A_z}{\partial_z}\right] d_x d_y d_z$$

That is $d_v = d_x d_y d_z$

$$\iiint_v \nabla . A \, dv = \iiint_v \left[\frac{\partial A_x}{\partial_x} d_x d_y d_z + \frac{\partial A_y}{\partial_y} d_x d_y d_z + \frac{\partial A_z}{\partial_z} d_x d_y d_z\right]$$

$$\iiint_v \nabla . A\, dv = \iiint_v \left[\frac{\partial A_x}{\partial_x} d_x d_y d_z\right] + \iiint_v \left[\frac{\partial A_y}{\partial_y} d_x d_y d_z\right] + \iiint_v \left[\frac{\partial A_z}{\partial_z} d_x d_y d_z\right]$$

Considered the first portion of equation

$$\iiint_v \frac{\partial A_x}{\partial_x} d_x d_y d_z$$

Assume the limit as $x_1\ to\ x_2$

$$\iiint_v \frac{\partial A_x}{\partial_x} d_x d_y d_z = \iiint_{x1}^{x_2} \frac{\partial A_x}{\partial_x} d_x d_y d_z$$

$$= \iint [A_x]_{x1}^{x2}\ d_y d_z$$

$$\iiint_{x1}^{x_2} \frac{\partial A_x}{\partial_x} d_x d_y d_z = \iint [A_{x2} - A_{x1}]\, d_y d_z$$

$$\iiint_v \frac{\partial A_x}{\partial_x} d_x d_y d_z = \iint [A_{x2} - A_{x1}]\, d_y d_z$$

Considered the second portion of equation

$$\iiint_v \left[\frac{\partial A_y}{\partial_y} d_x d_y d_z\right]$$

$$\iiint_v \left[\frac{\partial A_y}{\partial_y} d_x d_y d_z\right] = \iiint_{y1}^{y_2} \frac{\partial A_y}{\partial_y} d_x d_y d_z$$

$$= \iint [A_y]_{y1}^{y2}\ d_x d_z$$

$$\iiint_{y1}^{y_2} \frac{\partial A_y}{\partial_y} d_x d_y d_z = \iint [A_{y2} - A_{y1}]\, d_x d_z$$

$$\iiint_v \left[\frac{\partial A_y}{\partial_y} d_x d_y d_z\right] = \iint [A_{y2} - A_{y1}]\, d_x d_z$$

Considered the third portion of equation

$$\iiint_v \left[\frac{\partial A_z}{\partial_z} d_x d_y d_z\right]$$

$$\iiint_v \left[\frac{\partial A_z}{\partial_z} d_x d_y d_z\right]=\iiint_{z1}^{z_2} \frac{\partial A_z}{\partial_z} d_x d_y d_z$$

$$\iiint_{z1}^{z_2} \frac{\partial A_z}{\partial_z} d_x d_y d_z= \iint [A_z]_{z1}^{z2} \ \ d_x d_y$$

$$\iiint_v \left[\frac{\partial A_z}{\partial_z} d_x d_y d_z\right] = \iint [A_{z2} \ - A_{z1}] \, d_x d_y$$

Substitute the equation in below equation

$$\iiint_v \nabla . A \, dv = \iiint_v \left[\frac{\partial A_x}{\partial_x} d_x d_y d_z\right] + \iiint_v \left[\frac{\partial A_y}{\partial_y} d_x d_y d_z\right] + \iiint_v \left[\frac{\partial A_z}{\partial_z} d_x d_y d_z\right]$$

$$\iiint_v \nabla . A \, dv == \iint [A_{x2} - A_{x1}] \, d_y d_z + \iint [A_{y2} \ - A_{y1}] \, d_x d_z + \iint [A_{z2} \ - A_{z1}] \, d_x d_y$$

Assume $A_{x2} - A_{x1} \ = A_x$

$$A_{y2} - A_{y1} \ = A_y$$

$$A_{z2} - A_{z1} \ = A_z$$

Then $d_x d_y = ds_z$

$$d_y d_z = ds_x$$

$$d_z d_x = ds_y$$

Sub all these in above equation

$$\iiint_v \nabla . A \, dv == \iint [A_x] \, ds_x + \iint [A_y] \, ds_y + \iint [A_z] \, ds_z$$

$$\iiint_v \nabla . A \, dv = \iint [A_x + A_y + A_z] \, d_s$$

Assume $A_x + A_y + A_z = A$

Substitute $A_x + A_y + A_z = A$ in above equation

$$= \oiint_s A . ds$$

$$\iiint_v \nabla . A \, dv = \oiint_s A . ds$$

STROKES THEOREM:

Curl of a Vector

The curl of **A** is an axial (or) rotational vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

The curl of vector **A** at any point is defined as the limit of its surface integral of its cross product with normal over a closed surface per unit volume as the volume shrinks to zero.

$$|\mathbf{Curl}\ A| = \lim_{v \to 0} \frac{1}{v} \oiint_s \vec{n} \times A \, ds$$

It can expressed as

$$\nabla = \frac{\partial}{\partial_x} \overrightarrow{a_x} + \frac{\partial}{\partial_y} \overrightarrow{a_y} + \frac{\partial}{\partial_z} \overrightarrow{a_z}$$

$$A = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times A = \left(\frac{\partial A_z}{\partial_y} - \frac{\partial A_y}{\partial_z}\right)\overrightarrow{a_x} - \left(\frac{\partial A_z}{\partial_x} - \frac{\partial A_x}{\partial_z}\right)\overrightarrow{a_y} + \left(\frac{\partial A_y}{\partial_x} - \frac{\partial A_x}{\partial_z}\right)\overrightarrow{a_z}$$

$$\nabla \times A = \left(\frac{\partial A_z}{\partial_y} - \frac{\partial A_y}{\partial_z}\right)\overrightarrow{a_x} + \left(\frac{\partial A_x}{\partial_z} - \frac{\partial A_z}{\partial_x}\right)\overrightarrow{a_y} + \left(\frac{\partial A_y}{\partial_x} - \frac{\partial A_x}{\partial_z}\right)\overrightarrow{a_z}$$

This operation is called curl.

$$\nabla \times A = Curl\, A$$

STROKES THEOREM:

Strokes's theorem states that the circulation of a vector field $\boldsymbol{A}$ around a (closed) path $\boldsymbol{L}$ is equal to the surface integral of the curl of $\boldsymbol{A}$ over the open surface $\boldsymbol{S}$ bounded by $\boldsymbol{L}$**,** provided $\boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$ are continuous on $\boldsymbol{S}$**.**

The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any closed surface.

$$\oint H.dl = \iint_s^0 \nabla \times H\, ds$$

Proof:

Consider an arbitrary surface this is broken up into incremental surface of area $\Delta \boldsymbol{s}$ shown in figure. 1.4.2

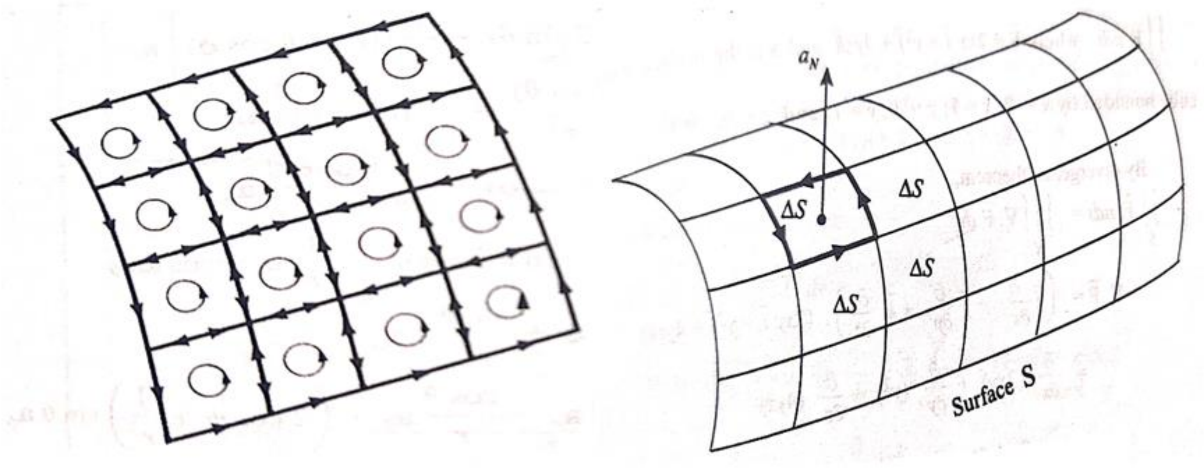

Figure 1.4.2 Illustration of Stroke's theorem

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-81]

If $\boldsymbol{H}$ is any field vector, then by definition of the curl to one of these incremental surfaces.

$$\frac{\oint \boldsymbol{H.dl\Delta s}}{\boldsymbol{\Delta s}} = (\boldsymbol{\nabla \times H})_{\boldsymbol{N}}$$

Where $\boldsymbol{N}$ indicates normal to the surface and $\boldsymbol{dl\Delta s}$ indicate that the closed path of an incremental area $\boldsymbol{\Delta s}$

The curl of $\boldsymbol{H}$ normal to the surface can be written as

$$\frac{\oint \boldsymbol{H.dl\Delta s}}{\boldsymbol{\Delta s}} = (\boldsymbol{\nabla \times H}) \,.\, \boldsymbol{a_N}$$

$$\oint \boldsymbol{H.\,dl\,\Delta s} = (\boldsymbol{\nabla \times H}) \,.\, \boldsymbol{a_N\,\Delta s}$$

$$\oint \boldsymbol{H.\,dl\,\Delta s} = (\boldsymbol{\nabla \times H}) \,.\, \boldsymbol{\Delta s}$$

Where $\boldsymbol{a_N}$ is the unit vector normal to $\boldsymbol{\Delta s}$

The closed integral for whole surface $\boldsymbol{S}$ is given by the surface integral of the normal component of curl $\boldsymbol{H}$

$$\oint \boldsymbol{H.\,dl} = \iint_{\boldsymbol{s}} \boldsymbol{\nabla \times H\; ds}$$