Two Point Sources with Currents Equal in Magnitude and Phase

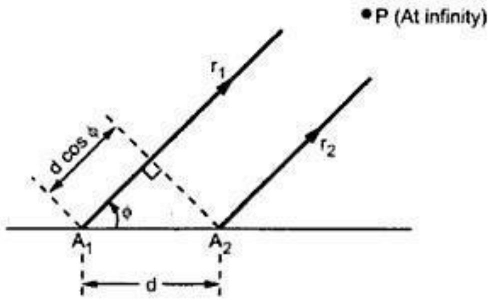

Fig. 5 Two element array

Consider two point sources A_1 and A_2, separated by distance d as shown in the Fig. 5. Consider that both the point sources are supplied with currents equal in magnitude and phase. Consider point P far away from the array. Let the distance between point P and point sources A_1 and A_2 be r_1 and r_2 respectively. As these radial distances are extremely large as compared with the distance of separation between two point sources i.e. d, we can assume, $r_1 = r_2 = r$

The radiation from the point source A_2 will reach earlier at point P than that from point source A_1 because of the path difference. The extra distance is travelled by the radiated wave from point source A_1 than that by the wave radiated from point source A_2.

Hence path difference is given by,
Path difference = d cos υ

The path difference can be expressed in terms of wavelength
as, Path difference = (d cos υ) / λ...**(2)**
Hence the phase angle υ is given by,

Phase angle υ = 2π (Path difference)

$$\therefore \quad \psi = 2\pi\left(\frac{d\cos\phi}{\lambda}\right)$$

$$\therefore \quad \boxed{\psi = \frac{2\pi}{\lambda} d\cos\phi \text{ rad}} \quad \ldots(3)$$

But phase shift β = 2π/λ, thus equation (3) becomes,

$$\therefore \quad \boxed{\psi = \beta d\cos\phi \text{ rad}} \quad \ldots(4)$$

Let E_1 be the far field at a distant point P due to point source AI. Similarly let E_2 be the far field at point P due to point source A_2. Then the total field at point P be the addition of the two field components due to the point sources A_1 and A_2. If the phase angle between the two fields is $\upsilon = \beta d \cos\upsilon$ then the far field component at point P due to point source A_1 is given by,

$$E_1 = E_0 \cdot e^{-j\frac{\Psi}{2}} \qquad \ldots(5)$$

Similarly the far field component at point P due to the point source A_2 is given by,

$$E_2 = E_0 \cdot e^{j\frac{\Psi}{2}} \qquad \ldots(6)$$

Note that the amplitude of both the field components is E_0 as currents are same and the point sources are identical. The total field at point P is given by,

$$E_T = E_1 + E_2 = E_0 \cdot e^{-j\frac{\Psi}{2}} + E_0 \cdot e^{j\frac{\Psi}{2}}$$

$$\therefore \quad E_T = E_0\left(e^{-j\frac{\Psi}{2}} + e^{j\frac{\Psi}{2}}\right)$$

Rearranging the terms on R.H.S., we get,

$$\therefore \quad E_T = 2E_0\left(\frac{e^{j\frac{\Psi}{2}} + e^{-j\frac{\Psi}{2}}}{2}\right) \qquad \ldots(7)$$

By trigonometric identity,

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta.$$

Hence equation (7) can be written as,

$$E_T = 2E_0 \cos\left(\frac{\Psi}{2}\right) \qquad \ldots(8)$$

Substituting value of Ψ from equation (4), we get,.

$$\therefore \quad \boxed{E_T = 2E_0 \cos\left(\frac{\beta d \cos\phi}{2}\right)} \qquad \ldots(9)$$

Above equation represents total field in intensity at point P. due to two point sources having currents of same amplitude and phase. The total amplitude of the field at point P is $2E_0$ while the phase shift is $\beta d\cos\upsilon/2$

The array factor is the ratio of the magnitude of the resultant field to the magnitude of the maximum field.

$$\therefore \quad A.F. = \frac{|E_T|}{|E_{max}|}$$

But maximum field is Ernax =2E0

$$\therefore \quad \boxed{A.F. = \frac{|E_T|}{|2\,E_0|} = \cos\left(\pi \frac{d}{\lambda} \cos\phi\right)}$$

The array factor represents the relative value of the field as a function of υ defines the radiation pattern in a plane containing the line of the array.

Maxima direction

From equation (9), the total field is maximum when $\cos\left(\frac{\beta d \cos\phi}{2}\right)$ is maximum. As we know, the variation of cosine of a angle is ± 1. Hence the condition for maxima is given by,

$$\boxed{\cos\left(\frac{\beta d \cos\phi}{2}\right) = \pm 1}$$

Let spacing between the two point sources be λ/2. Then we can write,

$$\cos\left[\frac{\beta(\lambda/2)\cos\phi}{2}\right] = \pm 1 \qquad \ldots(10)$$

$$\text{i.e. } \cos\left[\frac{\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\cos\phi}{2}\right] = \pm 1 \qquad \ldots \because \beta = \frac{2\pi}{\lambda}$$

$$\text{i.e. } \cos\left(\frac{\pi}{2}\cos\phi\right) = \pm 1$$

$$\text{i.e. } \frac{\pi}{2}\cos\phi_{max} = \cos^{-1}(\pm 1) = \pm n\pi, \quad \text{where } n = 0, 1, 2, \ldots\ldots$$

If n = 0, then

$$\frac{\pi}{2}\cos\phi_{max} = 0$$

i.e. $\cos\phi_{max} = 0$

i.e. $\boxed{\phi_{max} = 90° \text{ or } 270°}$ ...(11)

Minima direction

Again from equation (9), total field strength is minimum when $\cos\left(\frac{\beta d\cos\phi}{2}\right)$ is minimum i.e.

0 as cosine of angle has minimum value 0. Hence the condition for minima is given by,

$$\therefore \boxed{\cos\left(\frac{\beta d\cos\phi}{2}\right) = 0} \quad ...(12)$$

Again assuming d = λ/2 and β=2π/λ, we can write

$$\cos\left(\frac{\pi}{2}\cos\phi_{min}\right) = 0$$

$$\therefore \frac{\pi}{2}\cos\phi_{min} = \cos^{-1} 0 = \pm(2n+1)\frac{\pi}{2}, \text{ where } n = 0, 1, 2, ...$$

If n = 0, then,

$$\frac{\pi}{2}\cos\phi_{min} = \pm\frac{\pi}{2}$$

i.e. $\cos\phi_{min} = \pm 1$

i.e. $\boxed{\phi_{min} = 0° \text{ or } 180°}$ ...(13)

Half power point direction:

When the power is half, the voltage or current is 1/√2 times the maximum value.

Hence the condition for half power point is given by,

$$\boxed{\cos\left(\frac{\beta d\cos\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}} \quad ...(14)$$

Let d=λ/2 and β=2π/λ, then we can write,

$$\cos\left(\frac{\pi}{2}\cos\phi\right) = \pm\frac{1}{\sqrt{2}}$$

$$\text{i.e. } \frac{\pi}{2}\cos\phi = \cos^{-1}\left(\pm\frac{1}{\sqrt{2}}\right) = \pm(2n+1)\frac{\pi}{4}, \text{ where } n = 0, 1, 2, \ldots$$

If n = 0, then

$$\frac{\pi}{2}\cos\phi_{HPPD} = \pm\frac{\pi}{4}$$

$$\text{i.e. } \cos\phi_{HPPD} = \pm\frac{1}{2}$$

$$\text{i.e. } \phi_{HPPD} = \cos^{-1}\left(\pm\frac{1}{2}\right)$$

$$\therefore \quad \boxed{\phi_{HPPD} = 60^\circ \text{ or } 120^\circ} \quad \ldots(15)$$

The field pattern drawn with ET against υ for d=λ/2, then the pattern is bidirectional as shown in Fig 6. The field pattern obtained is bidirectional and it is a figure of eight.

If this pattern is rotated by 360^0 about axis, it will represent three dimensional doughnut shaped space pattern. This is the simplest type of broadside array of two point sources and it is called Broadside couplet as two radiations of point sources are in phase.

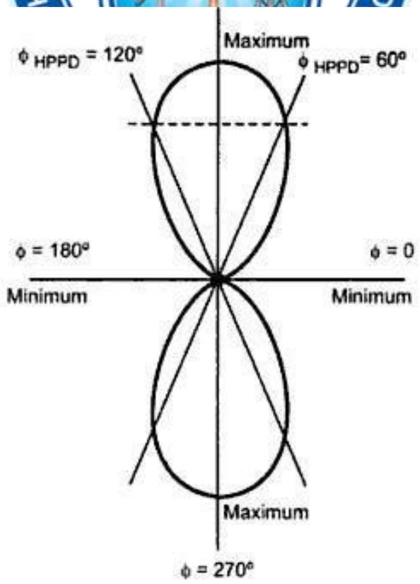

Fig. 6 Field pattern for two point source with spacing d=λ/2 and fed with currents equal in magnitude andphase.

<u>Two Point Sources with Currents Equal in Magnitudes but Opposite in Phase</u>

Consider two point sources separated by distance d and supplied with currents equal in magnitude but opposite in phase. Consider Fig. 5 all the conditions are exactly same except the phase of the currents is opposite i.e. 180°. With this condition, the total field at far point P is given by,

$$E_T = (-E_1) + (E_2) \quad \ldots(1)$$

Assuming equal magnitudes of currents, the fields at point P due to the point sources A1 and A2 can be written as,

$$E_1 = E_0 e^{-j\frac{\psi}{2}} \quad \ldots(2)$$

and

$$E_2 = E_0 e^{j\frac{\psi}{2}} \quad \ldots(3)$$

Substituting values of E1 and E2 in equation (1), we get

$$E_T = -E_0 \cdot e^{-j\frac{\psi}{2}} + E_0 \cdot e^{j\frac{\psi}{2}}$$

$$\therefore \quad E_T = E_0\left(-e^{-j\frac{\psi}{2}} + e^{j\frac{\psi}{2}}\right)$$

Rearranging the terms in above equation, we get,

$$\therefore \quad E_T = (j2)\, E_0\left(\frac{e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}}{j2}\right) \quad \ldots(4)$$

By trigonometry identity,

$$\frac{e^{j\theta} - e^{-j\theta}}{2} = \sin\frac{\theta}{2}.$$

Equation (4) can be written as,

$$E_T = j2E_0 \sin\left(\frac{\psi}{2}\right) \quad \ldots(5)$$

Now as the condition for two point sources with currents in phase and out of phase is exactly same, the phase angle can be written as previous case.

Phase angle = βdcosυ ...(6) Substituting value of phase angle in equation (5), we get,

$$\boxed{E_T = j(2E_0)\sin\left(\frac{\beta d\cos\phi}{2}\right)} \quad \ldots(7)$$

Maxima direction

From equation (7), the total field is maximum when $\sin\left(\frac{\beta d\cos\phi}{2}\right)$ is maximum i.e. ±1 as the maximum value of sine of angle is ±1. Hence condition for maxima is given by,

$$\boxed{\sin\left(\frac{\beta d\cos\phi}{2}\right) = \pm 1} \qquad \ldots(8)$$

Let the spacing between two isotropic point sources be equal to $d=\lambda/2$

Substituting $d=\lambda/2$ and $\beta=2\pi/\lambda$, in equation (8), we get,

$$\sin\left(\frac{\pi}{2}\cos\phi\right) = \pm 1$$

$$\text{i.e.} \quad \frac{\pi}{2}\cos\phi = \pm(2n+1)\frac{\pi}{2}, \text{ where } n = 0, 1, 2, \ldots\ldots$$

If n = 0. then

$$\frac{\pi}{2}\cos\phi_{max} = \pm\frac{\pi}{2}$$

$$\text{i.e. } \cos\phi_{max} = \pm 1$$

$$\text{i.e.} \quad \boxed{\phi_{max} = 0^\circ \text{ and } 180^\circ} \qquad \ldots(9)$$

Minima direction

Again from equation (7), total field strength is minimum when $\sin\left(\frac{\beta d\cos\phi}{2}\right)$ is minimum i.e. 0.

Hence the condition for minima is given by,

$$\boxed{\sin\left(\frac{\beta d\cos\phi}{2}\right) = 0} \qquad \ldots(10)$$

Assuming $d=\lambda/2$ and $\beta=2\pi/\lambda$ in equation (10), we get,

$$\sin\left(\frac{\pi}{2}\cos\phi\right) = 0$$

$$\text{i.e.} \quad \frac{\pi}{2}\cos\phi = \pm n\pi, \text{ where } n = 0, 1, 2, \ldots\ldots$$

If n = 0, then

$$\frac{\pi}{2}\cos\phi_{min} = 0$$

$$\text{i.e.} \quad \cos\phi_{min} = 0$$

$$\text{i.e.} \quad \boxed{\phi_{min} = +90° \text{ or } -90°} \qquad \ldots(11)$$

Half Power Point Direction (HPPD)

When the power is half of maximum value, the voltage or current equals to $1/\sqrt{2}$ times the respective maximum value. Hence the condition for the half power point can be obtained from equation (7) as,

$$\boxed{\sin\left(\frac{\beta d\cos\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}} \qquad \ldots(12)$$

Let $d=\lambda/2$ and $\beta=2\pi/\lambda$, we can write,

$$\sin\left(\frac{\pi}{2}\cos\phi\right) = \pm\frac{1}{\sqrt{2}}$$

$$\text{i.e.} \quad \frac{\pi}{2}\cos\phi = \pm(2n+1)\frac{\pi}{4}, \quad \text{where } n = 0, 1, 2.$$

If n = 0, we can write,

$$\frac{\pi}{2}\cos\phi_{HPPD} = \pm\frac{\pi}{4}$$

$$\text{i.e.} \quad \cos\phi_{HPPD} = \pm\frac{1}{2}$$

$$\therefore \quad \boxed{\phi_{HPPD} = 60° \text{ or } 120°} \qquad \ldots(13)$$

Thus from the conditions of maxima, minima and half power points, the field pattern can be drawn as shown in the Fig. 7.

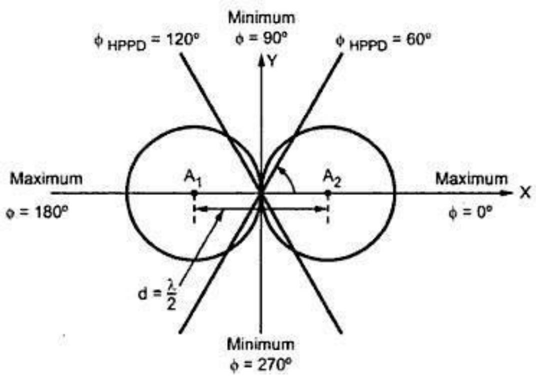

Fig. 7 Field pattern for two point sources with spacing d = d=λ/2 and fed with currents equal in magnitude but out of phase by 180^0.

As compared with the field pattern for two point sources with inphase currents, the maxima have shifted by 90° along X-axis in case of out-phase currents in two point source array. Thus the maxima are along the axis of the array or along the line joining two point sources. In first case, we have obtained vertical figure of eight. Now in above case, we have obtained horizontal figure of eight. As the maximum field is along the line joining the two point sources, this is the simple type of the end fire array.

Two point sources with currents unequal in magnitude and with any phase

Let us consider Fig. 5. Assume that the two point sources are separated by distance d and supplied with currents which are different in magnitudes and with any phase difference say α. Consider that source 1 is assumed to be reference for phase and amplitude of the fields E_1 and E_2, which are due to source 1 and source 2 respectively at the distant point P. Let us assume that E_1 is greater than E_2 in magnitude as shown in the vector diagram in Fig. 8.

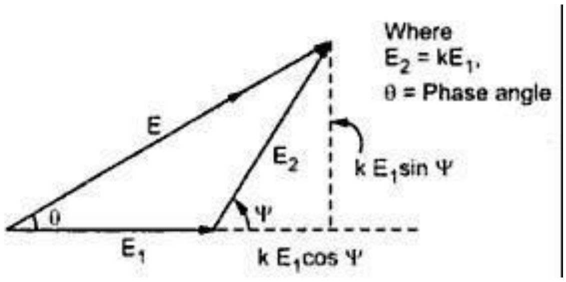

Fig. 8 Vector diagram of fields E_1 and E_2

Now the total phase difference between the radiations by the two point sources at any far point P is given by,

$$\Psi = \frac{2\pi}{\lambda}\cos\phi + \alpha \qquad \text{... (1)}$$

where α is the phase angle with which current I2 leads current II. Now if α = 0, then the condition is similar to the two point sources with currents equal in magnitude and phase. Similarly if α = 180", then the condition is similar to the two point source with currents equal in magnitude but opposite in phase. Assume value of phase

0 difference as 0 <α< 180 . Then the resultant field at point P is given by,

$$E_T = E_1 e^{j0} + E_2 e^{j\psi}$$ ...(source 1 is assumed to be reference hence phase angle is 0)

$$\therefore \quad E_T = E_1 + E_2 e^{j\psi}$$

$$\therefore \quad E_T = E_1\left(1 + \frac{E_2}{E_1} e^{j\psi}\right)$$

$$\text{Let} \quad \frac{E_2}{E_1} = k \qquad \text{...(2)}$$

Note that E1> E2, the value of k is less than unity. Moreover the value of k is given by, 0 ≤ k ≤ 1

$$\therefore \quad E_T = E_1 [1 + k(\cos\psi + j\sin\psi)] \qquad \text{... (3)}$$

The magnitude of the resultant field at point P is given by,

$$|E_T| = |E_1 [1 + k\cos\psi + jk\sin\psi]|$$

$$\therefore \quad |E_T| = E_1\sqrt{(1 + k\cos\psi)^2 + (k\sin\psi)^2} \qquad \text{... (4)}$$

The phase angle between two fields at the far point P is given by,

$$\therefore \quad \theta = \tan^{-1}\frac{k\sin\psi}{1 + k\cos\psi} \qquad \text{... (5)}$$