Taylor's Series for Functions of Two Variables

Let $f(x,y)$ be a function of two variables x,y. We can expand $f(x+h,y+k)$ in a series of ascending powers of h and k. Consider $f(x+h,y+k)$ as a function of the single variable x.

The Taylors series is

$$\boldsymbol{f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +}$$

$$\boldsymbol{\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots}$$

It is the Taylor's series expansion of $f(x,y)$ about the point (a,b).

Example:

(i) Expand $e^x cosy$ about $\left(0,\frac{\pi}{2}\right)$ up to the third term using Taylor's series.

(ii) $e^x cosy$ in powers of x and y as far as the terms of the third degree .

Solution:

Function	Value at $\left(0,\frac{\pi}{2}\right)$	Value at (0, 0)
$f(x,y) = e^x cosy$	$f = 0$	1
$f_x = e^x cosy$ $f_y = -e^x siny$	$f_x = 0$ $f_y = -1$	1 0
$f_{xx} = e^x cosy$ $f_{xy} = -e^x \text{siny}$ $f_{yy} = -e^x \text{cosy}$	$f_{xx} = 0$ $f_{xy} = -1$ $f_{yy} = 0$	1 0 -1
$f_{xxx} = e^x cosy$ $f_{xxy} = -e^x \text{siny}$ $f_{xyy} = -e^x cosy$ $f_{yyy} = e^x \sin y$	$f_{xxx} = 0$ $f_{xxy} = -1$ $f_{xyy} = 0$ $f_{yyy} = 1$	1 0 -1 0

By Taylor's theorem

$f(x,y) = f\ (a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-\right.$

$\left.b)^2 f_{xyy}(a,b) + \qquad (y-b)^3 f_{yyy}(a,b)\right] + \cdots$

(i) $\boldsymbol{a = 0, b = \frac{\pi}{2}}$

$f\ (x,y) = 0 + \frac{1}{1!}\left[(x)(0) + \left(y-\frac{\pi}{2}\right)(-1)\right] + \frac{1}{2!}\left[(x)^2(0) + 2(x)(y-\frac{\pi}{2})(-1) + \right.$

$\left.\left(y-\frac{\pi}{2}\right)^2(0)\right]$

$+\frac{1}{3!}\left[(x)^3(0) + 3(x)^2\left(y-\frac{\pi}{2}\right)(-1) + 3(x)\left(y-\frac{\pi}{2}\right)^2(0) + \left(y-\frac{\pi}{2}\right)^3(1)\right] + \cdots$

$= -y + \frac{\pi}{2} + \frac{1}{2!}\left[-2xy + 2x\frac{\pi}{2}\right] + \frac{1}{3!}\left[-3x^2y + 3\frac{\pi}{2}x^2 + \left(y-\frac{\pi}{2}\right)^3\right]$

(ii) $\boldsymbol{a = 0, b = 0}$

$f\ (x,y) = 1 + \frac{1}{1!}[(x)(1) + (y)(0)] + \frac{1}{2!}[(x)^2(1) + 2(x)(y)(0) + (y)^2(-1)]$

$+\frac{1}{3!}[(x)^3(1) + 3(x)^2(y)(0) + 3(x)(y)^2(-1) + (y)^3(0)] + \cdots$

$f\ (x,y) = 1 + x + \frac{1}{2!}[x^2 - y^2] + \frac{1}{3!}[x^3 - 3xy^2] + \cdots$

Example:

Obtain terms up to the third degree in the Taylor series expansion of $\boldsymbol{e^x siny}$ **about the point** $\left(\boldsymbol{1}, \boldsymbol{\frac{\pi}{2}}\right)$

Solution:

Function	**Value at** $(1, \frac{\pi}{2})$
$f\ (x,y) = e^x siny$	$f = e$
$f_x = e^x siny$ $f_y = e^x cosy$	f_x = e f_y = 0
$f_{xx} = e^x siny$ $f_{xy} = e^x \cos y$ $f_{yy} = -e^x \sin y$	f_{xx} = e f_{xy} = 0 $f_{yy} = -e$

$f_{xxx} = e^x siny$ $f_{xxy} = e^x \cos y$ $f_{xyy} = -e^x siny$ $f_{yyy} = -e^x cosy$	$f_{xxx} = e$ $f_{xxy} = 0$ $f_{xyy} = -e$ $f_{yyy} = 0$

By Taylor's theorem

$$f(x,y) = f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\cdots$$

Put $\boldsymbol{a = 1, b = \frac{\pi}{2}}$

$$f(x,y) = e+\frac{1}{1!}\left[(x-1)e+\left(y-\frac{\pi}{2}\right)(0)\right]+$$
$$\frac{1}{2!}\left[(x-1)^2 e+2(x-1)\left(y-\frac{\pi}{2}\right)(0)+\left(y-\frac{\pi}{2}\right)^2(-e)\right]+$$
$$\frac{1}{3!}\left[(x-1)^3 e+3(x-1)^2\left(y-\frac{\pi}{2}\right)(0)+3(x-1)\left(y-\frac{\pi}{2}\right)^2(-e)+\left(y-\frac{\pi}{2}\right)^3(0)\right]+\cdots$$

$$f(x,y) = e+\frac{1}{1!}(x-1)e+\frac{1}{2!}\left[(x-1)^2 e+\left(y-\frac{\pi}{2}\right)^2(-e)\right]$$
$$+\frac{1}{3!}\left[(x-1)^3 e-3e(x-1)\left(y-\frac{\pi}{2}\right)^2\right]+\cdots$$

Example:

Expand the function $\boldsymbol{sin\,xy}$ in powers of $\boldsymbol{x-1}$ and $\boldsymbol{y-\frac{\pi}{2}}$ upto second degree terms.

Solution:

Function	**Value at $(1,\frac{\pi}{2})$**
$f(x,y) = sinxy$	$f = 1$
$f_x = ycos(xy)$ $f_y = xcos(xy)$	$f_x = 0$ $f_y = 0$

$f_{xx} = -y^2\sin(xy)$ $f_{xy} = -xy\sin(xy)+\cos(xy)$ $f_{yy} = -x^2\sin xy$	$f_{xx} = -\frac{\pi^2}{4}$ $f_{xy} = -\frac{\pi}{2}$ $f_{yy} = -1$

By Taylor's theorem

$$f(x,y) = f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+\cdots$$

Put $a = 1, b = \frac{\pi}{2}$

$$f(x,y) = 1+\frac{1}{1!}\left[(x-1)(0)+\left(y-\frac{\pi}{2}\right)(0)\right]+$$
$$\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2(-1)\right]+\cdots$$
$$= 1+\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right)-\left(y-\frac{\pi}{2}\right)^2\right]+\cdots$$
$$= 1+\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right)-\pi(x-1)\left(y-\frac{\pi}{2}\right)-\left(y-\frac{\pi}{2}\right)^2\right]+\cdots$$

Example:

Expand $f(x,y) = e^{xy}$ in Taylors Series at (1, 1) upto second degree.

Solution:

Function	Value at (1,1)
$f(x,y) = e^{xy}$	$f = e$
$f_x = y\,e^{xy}$ $f_y = x\,e^{xy}$	$f_x = e$ $f_y = e$
$f_{xx} = y^2 e^{xy}$ $f_{xy} = x\,y\,e^{xy}+e^{xy}$ $f_{yy} = x^2 e^{xy}$	$f_{xx} = e$ $f_{xy} = e + e = 2e$ $f_{yy} = e$

By Taylor's theorem

$$f(x,y) = f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+\cdots$$

Put $a = 1, b = 1$

$f\ (x,y) = e + \frac{1}{1!}[(x-1)e + (y-1)(e)] +$

$$\frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)(2e) + (y-1)^2(e)] + \cdots$$

Example:

Expand $e^x \log(1+y)$ in powers of x and y upto terms of third degree.

Solution:

Function	**Value at (0 ,0)**
$f\ (x,y) = e^x \log(1+y)$	$f = 0$
$f_x = e^x \log(1+y)$ $f_y = e^x \frac{1}{1+y}$	$f_x = 0$ $f_y = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x \frac{1}{1+y}$ $f_{yy} = -e^x \frac{1}{(1+y)^2}$	$f_{xx} = 0$ $f_{xy} = 1$ $f_{yy} = -1$
$f_{xxx} = e^x \log(1+y)$ $f_{xxy} = e^x \frac{1}{1+y}$ $f_{xyy} = -e^x \frac{1}{(1+y)^2}$ $f_{yyy} = 2\, e^x \frac{1}{(1+y)^3}$	$f_{xxx} = 0$ $f_{xxy} = 1$ $f_{xyy} = -1$ $f_{yyy} = 2$

By Taylor's theorem

$f\ (x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$

$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\cdots$

Put $\boldsymbol{a = 0, b = 0}$

$f(x,y) = 0+\frac{1}{1!}[(x)(0)+(y)(1)]+\frac{1}{2!}[(x)^2(0)+2(x)(y)(1)+(y)^2(-1)]$

$+\frac{1}{3!}[(x)^3(0)+3(x)^2(y)(1)+3(x)(y)^2(-1)+(y)^3(2)]+\cdots$

$= y+\frac{2xy-y^2}{2!}+\frac{3x^2y-3xy^2+2y^3}{3!}+\cdots$

Example:

Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ up to the third degree term

Solution:

Let $f(x,y) = x^2y+3y-2$

Function	**Value at (1, -2)**
$f(x,y) = x^2y+3y-2$	$f = -10$
$f_x = 2xy$ $f_y = x^2+3$	$f_x = -4$ $f_y = 4$
$f_{xx} = 2y$ $f_{xy} = 2x$ $f_{yy} = 0$	$f_{xx} = -4$ $f_{xy} = 2$ $f_{yy} = 0$
$f_{xxx} = 0$ $f_{xxy} = 2$ $f_{xyy} = 0$ $f_{yyy} = 0$	$f_{xxx} = 0$ $f_{xxy} = 2$ $f_{xyy} = 0$ $f_{yyy} = 0$

By Taylor's theorem

$f(x,y) = f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$

$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$

$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\cdots$

Put $a = 1, b = -2$

$f(x,y) = -10 + \frac{1}{1!}[(x-1)(-4)+(y+2)(4)]+$

$$\frac{1}{2!}[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)]$$

$+\frac{1}{3!}[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)]+\cdots$

$= -10 - 4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$