4.5 Homomorphism

Let $(G, \cdot)$ and $(G',*)$ be any two groups.

A mapping $f: G \to G'$ is said to be a homomorphism, if $f(a \cdot b) = f(a) * f(b)$ for any $a, b \in G$ is called a group homomorphism.

Example: (i)

Let $f: (Z, +) \to (Z, +)$ given by $f(x) = 2x \; \forall \; x \in Z$ is a homomorphism.

For, $x, y \in Z, f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y)$

Example: (ii)

Let $f: (R, +) \to (R^+, \cdot)$ given by $f(x) = e^x \; \forall \; x \in R$ is a homomorphism.

For, $x \in R, \; f(x + y) = e^{x+y} = e^x \cdot e^y = f(x) \cdot f(y)$

Isomorphism:

Let $(G, \cdot)$ and $(G', *)$ be any two groups. A mapping $f: G \to G'$ is said to be isomorphism if

(i) f is one – one

(ii) f is onto

(iii) f is homomorphism

Types of Homomorphism

(i) If f is one – to – one then f is monomorphism.

(ii) (ii) If f is onto then f is epimorphism.

Theorem: 1

Homomorphism preserves identities.

Proof:

Let $a \in G$

Let f be a homomorphism from $(G, *)$ and $(G', *)$

Clearly $f(a) \in G'$

$\Rightarrow f(a) * e' = f(a)$ $(e' - identity\ in\ G')$

$= f(a * e)$ (e – identity in G)

$= f(a) * f(e)$ (f – homomorphism)

$\Rightarrow\ e' = f(e)$ (Left cancellation law)

Hence f preserves identities.

Hence the proof.

Theorem: 2

Homomorphism preserves inverse.

Proof:

Let $a \in G$

Since G is a group, $a^{-1} \in G$

Since G is a group $a * a^{-1} = a^{-1} * a = e$

Consider $a * a^{-1} = e$

$$\Rightarrow f(a * a^{-1}) = f(e)$$

$$\Rightarrow f(a) * f(a^{-1}) = e' \because e' = f(e), f \text{ is homomorphism}$$

$\Rightarrow f(a^{-1})$ is the inverse of $f(a) \in G'$

Hence $[f(a)]^{-1} = f(a^{-1})$

Hence f preserves inverse.

Hence the proof.

Kernal of Homomorphism

Let $f: G \to G'$ be a group homomorphism. The set of elements of G which are mapped into e' (identity in G') is called the kernel of f and it is denoted by ker(f)

$$\ker(f) = \{x \in G / f(x) = e'\}$$

Theorem: 1

Kernel of a homomorphism of a group into another group is a normal subgroup.

Proof:

Let $(G,*)$ and $(G', \oplus)$ be two groups.

$f: (G,*) \rightarrow (G', \oplus)$ is a homomorphism.

Define $\ker(f) = \{x \in G \,/ f(x) = e'\}$

Claim: Ker f is a normal subgroup of G

We know that homomorphism preserves identity.

$i.e., f(e) = e'$, so $e \in kerf$

⇒Ker f is non empty.

(ii) $a, b \in \ker f \Rightarrow a * b^{-1} \in kerf$ then ker f is a subgroup.

$a \in kerf \Rightarrow f(a) = e'$ by definition of ker f

$b \in kerf \Rightarrow f(b) = e'$ by definition of ker f

Since homomorphism preserves inverse $\Rightarrow [f(a)]^{-1} = f(a^{-1})$

Now $f(a * b^{-1}) = f(a) \oplus f(b^{-1})$

$= f(a) \oplus [f(b)]^{-1}$

$= e' \oplus e'$

$= e'$

$\Rightarrow a * b^{-1} \in kerf$

Hence kerf is a subgroup of G.

(iii)Let $a \in kerf \Rightarrow f(a) = e'$ by definition of kerf

Homomorphism preserves inverses $\Rightarrow [f(a)]^{-1} = f(a^{-1})$

So $f(g^{-1} * a * g) = f(g^{-1}) \oplus f(a) \oplus f(g)$

$= [f(g)]^{-1} \oplus e' \oplus f(g)$

$= [f(g)]^{-1} \oplus f(g)$

$= e'$

Hence by definition, $g^{-1} * a * g \in kerf$

Hence kerf is a normal subgroup.

Hence the proof.

Theorem:2

Fundamental theorem of group homomorphism

Every homomorphic image of a group G is isomorphic to some quotient group of G.

(OR)

Let $f: G \rightarrow G'$ be a onto homomorphism of groups with kernel K, then $\frac{G}{K} \cong G'$

Proof:

Let f be the homomorphism $f: G \rightarrow G'$

Let G' be the homomorphic image of a group G.

Let K be the kernel of this homomorphism.

Clearly K is a normal subgroup of G.

Claim: $\frac{G}{K} \cong G'$

Define φ: $\frac{G}{K} \rightarrow G'$ by $\varphi(K * a) = f(a)$ for all $a \in G$

(i) φ is well defined.

We have $K * a = K * b$

$$\Rightarrow a * b^{-1} \in K$$

$\Rightarrow f(a * b^{-1}) = e'$ (e' is identity)

$\Rightarrow f(a\ ) * f(b^{-1}) = e'$

$\Rightarrow f(a) * [f(b)]^{-1} = e'$

$\Rightarrow f(a) * [f(b)]^{-1} \ * f(b) = e' * f(b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \varphi(K * a) = \varphi(K * b)$

Hence φ is well defined.

(ii) To prove φ is one – one.

To prove $\varphi(K * a) = \varphi(K * b) \Rightarrow K * a = \ K * b$

We know that $\varphi(K * a) = \varphi(K * b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f(a\ ) * f(b^{-1}) = f(b\ ) * f(b^{-1})$

$= f(b * b^{-1})$

$= f(e)$

$\Rightarrow f(a\ ) * f(b^{-1}) = e'$

$\Rightarrow f(a * b^{-1}) = e'$

$\Rightarrow a * b^{-1} \in K$

$\Rightarrow K\ * a * b^{-1} = K$

$\Rightarrow K * a = \ K * b$

Hence φ is one – one.

(iii) φ is onto.

Let $y \in G'$

Since f is onto, there exists $a \in G$ such that $f(a) = y$

Hence $\varphi(K * a) = f(a) = y$

Hence φ is onto.

(iv) φ is a homomorphism.

Now $\varphi(K * a \ * \ K * b) = \varphi(K * a \ * b)$

$$= f(a * b)$$

$$= f(a) * f(b)$$

$$= \varphi(K * a) * (K * b)$$

Hence φ is a homomorphism.

Since φ is one – one, onto, homomorphism φ is an isomorphism between $\frac{G}{K}$ and G'.

Hence $\frac{G}{K} \cong \ G'$

Hence the proof.