4.3 NYQUIST STABILITY CRITERION

Nyquist criterion is a graphical method of determining stability of feedback control systems by using the Nyquist plot of their open-loop transfer functions.

Feedback transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Poles and zeros of the open loop transfer function

$$G(s)H(s) = \frac{K(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

$$1 + G(s)H(s) = \frac{(s - p_1)(s - p_2)\dots(s - p_n) + K(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

Number of closed loop poles – Number of zeros of 1+GH = Number of open loop poles

$$1 + G(s)H(s) = \frac{(s - z_{c1})(s - z_{c2})\dots(s - z_{cm})}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

where, z_{c1}, z_{c2}, ….., z_{cm} = zeros of 1+G(s)H(s)

These are also poles of the closed loop transfer function

$$Magnitude, |1 + G(s)H(s)| = \frac{|(s - z_{c1})||(s - z_{c2})|\dots|(s - z_{cm})|}{|(s - p_1)||(s - p_2)|\dots|(s - p_n)|}$$

$$Angle, \angle 1 + G(s)H(s) = \frac{\angle(s - z_{c1})\angle(s - z_{c2})\dots\angle(s - z_{cm})}{\angle(s - p_1)\angle(s - p_2)\dots\angle(s - p_n)}$$

The s-plane to 1+GH plane mapping phase angle of the 1+G(s)H(s) vector, corresponding to a point on the s-plane is the difference between the sum of the phase of all vectors drawn from zeros of 1+GH (closed loop poles) and open loops on the s plane. If this point s is moved along a closed contour enclosing any or all of the above zeros and poles, only the phase of the vector of each of the enclosed zeros or open-loop poles will change by 3600. The direction will be in the same sense of the contour enclosing zeros and in the opposite sense for the contour enclosing open-loop poles. A stability test for time invariant linear systems can also be derived in the frequency domain. It is known as Nyquist stability criterion. It is based on the complex analysis result known as *Cauchy's principle of argument*. Note that the system transfer function is a complex function. By applying Cauchy's principle of argument to the *open-loop system* transfer function, we will get information about stability of the closed-loop system transfer function and arrive at the Nyquist stability criterion (Nyquist, 1932). The importance of Nyquist stability lies in the fact that it can also be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins. These stability margins are needed for frequency domain controller design techniques. Only the essence of the Nyquist stability criterion is presented and the phase and gain stability margins are defined. The Nyquist method is used for studying the stability of linear systems with pure time delay.

For a SISO feedback system the closed-loop transfer function is given by,

$$M(s) = \frac{G(s)}{1 + G(s)H(s)}$$

where, G(s) represents the system and H(s) is the feedback element. Since the system poles are determined as those values at which its transfer function becomes infinity, it follows that the closed-loop system poles are obtained by solving the following equation.

$$1 + G(s)H(s) = 0 = \Delta(s)$$

which, in fact, represents the *system characteristic equation*.

Principles of Argument

When a closed contour in the s-plane encloses a certain number of poles and zeros of 1+G(s)H(s) in the clockwise direction, the number of encirclements of the origin by the corresponding contour in the G(s)H(s) plane will encircle the point (-1,0) a number of times given by the difference between the number of its zeros and poles of 1+G(s)H(s) it enclosed on the s-plane. Let F(s) be an analytic function in a closed region of the complex plane given in figure 4.3.1 except at a finite number of points (namely, the poles of F(s)). It is also assumed that F(s) is analytic at everypoint on the contour. Then, *as s travels around the contour in the s - plane in the clockwise direction, the function encircles the origin in the (Re{F(s)}, Im{F(s)}) - plane in the same direction times (see figure 4.3.1), with given by,*

$$N = Z – P$$

where Z and P stand for the number of zeros and poles (including their multiplicities) of the function F(s) inside the contour.

$$arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

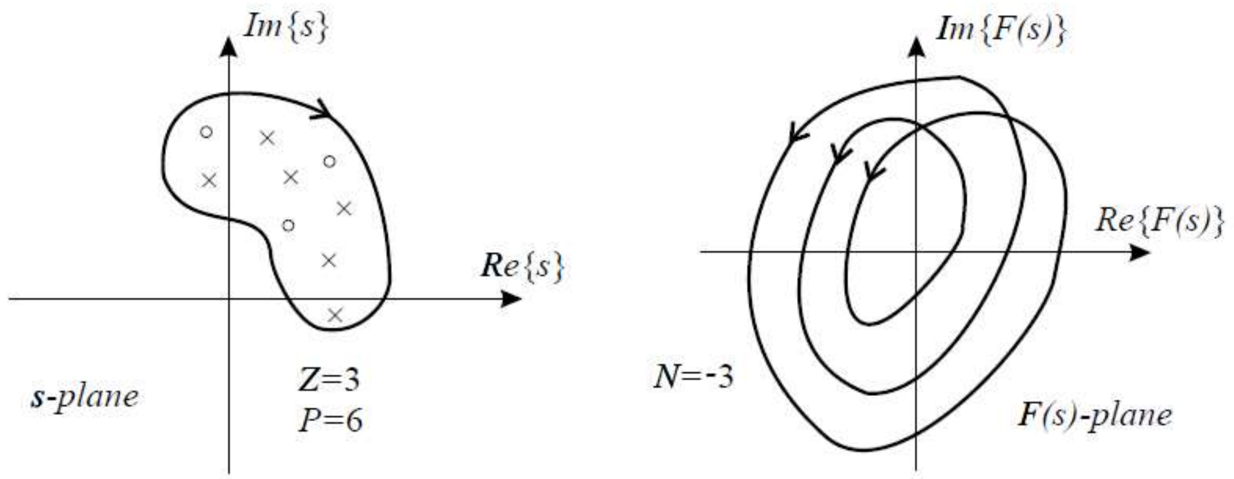

Figure 4.3.1 s-plane and F(s) plane contours

[Source: "Control Systems" by A Nagoor Kani, Page: 4.27]

Contour in the s-plane

The Nyquist plot is a polar plot of the function D(s) = 1+G(s)H(s) when 's' travels around the contour given in figure 4.3.2.

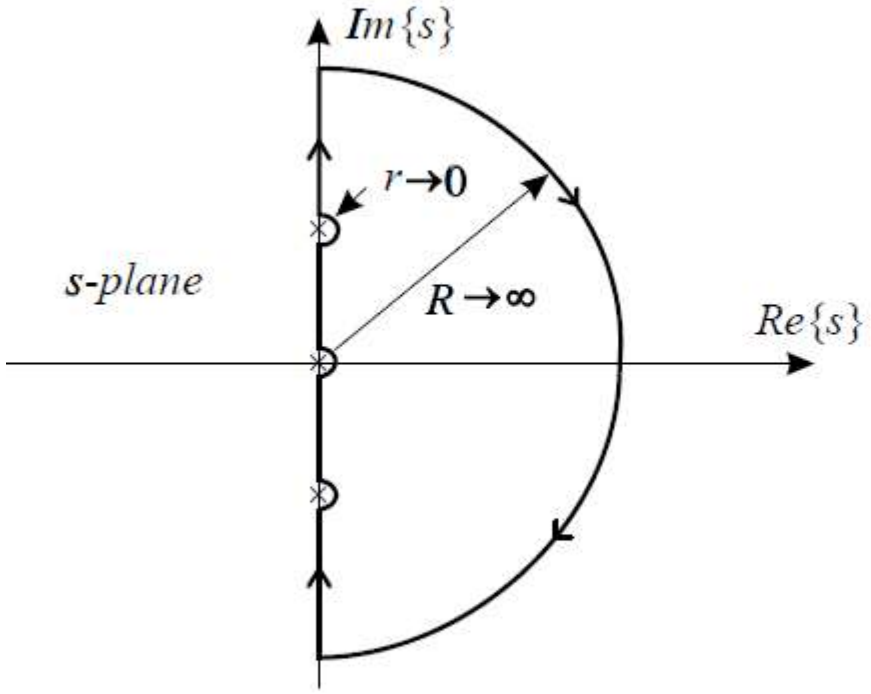

Figure 4.3.2 Nyquist contour when the poles are on imaginary axis and at origin

[Source: "Control Systems" by A Nagoor Kani, Page: 4.33]

Phase and Gain Stability Margins

Two important notions can be derived from the Nyquist diagram: *phase and gain stability margins*. The phase and gain stability margins are presented in figure 4.3.3.

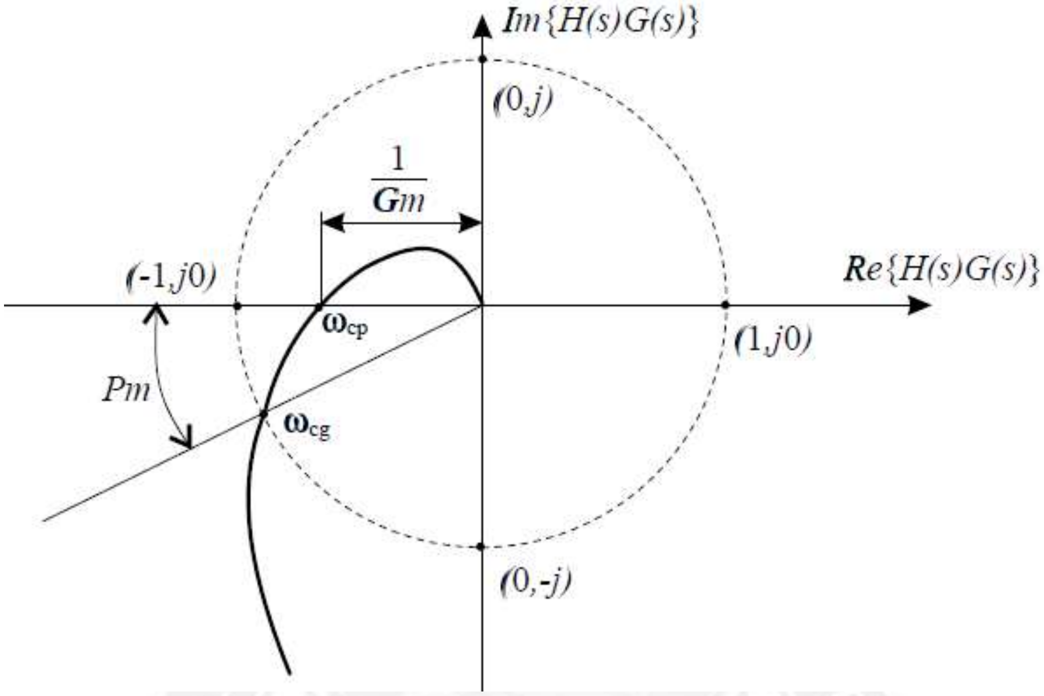

Figure 4.3.3 Gain and Phase margin

[Source: "Control Systems" by A Nagoor Kani, Page: 4.33]

They give the degree of relative stability; in other words, they tell how far the given system is from the instability region. Their formal definitions are given by

$$PM = 180^o + arg\{G(j\omega_{gc})H(j\omega_{gc})\}$$

$$GM(dB) = 20\log\frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}, (dB)$$

where, ω_{gc} and ω_{pc} stand for gain and phase crossover frequency respectively.

$$|G(j\omega_{gc})H(j\omega_{gc})| = 1 \Rightarrow \omega_{gc}$$

$$arg\{G(j\omega_{pc})H(j\omega_{pc})\} = 180^o \Rightarrow \omega_{pc}$$

PROCEDURE FOR INVESTIGATING STABILITY USING NYQUIST CRITERION

The following procedure can be followed to investigate the stability of closed loop system from the knowledge of open loop system, using Nyquist stability criterion.

1. Choose a Nyquist contour as shown in figure, which encloses the entire right half s-plane except the singular points. The Nyquist contour encloses all the right half s-plane poles and zeros of G(s)H(s). [The poles on imaginary axis are singular points and so they are avoided by taking a detour around it as shown in figures.
2. The Nyquist contour should be mapped in the G(s)H(s)-plane using the function G(s)H(s) to determine the encirclement -1 + j0 point in the G(s)H(s)-plane. The

Nyquist contour of the figure can be divided into four sections C_1.C_2.C_3 and C_4. The mapping of the four sections in the G(s)H(s)-plane can be carried sectionwise and then combined together to get entire G(s)H(s)-contour.

3. In section C_1, the value of ω varies from 0 to + infinite. The mapping of section C_1 is obtained by letting s = jω in G(s)H(s) and varying ω from 0 to + infinite.

   The locus of G(jω)H(jω) as ω is varied from 0 to + infinite will be the G(s)H(s)-contour in G(s)H(s)-plane corresponding to section C_1 in s-plane. This locus is the plot of G(jω)H(jω). There are three ways of mapping this section of G(s)H(s)-contour, they are,

   (i) Calculate the values of G(jω)H(jω) for various values of ω and sketch the actual locus of G(jω)H(jω).

   (or)

   (ii) Separate the real part and imaginary part of G(jω)H(jω). Equate the imaginary part to zero, to find the frequency at which the G(jω)H(jω) locus crosses real axis (to find phase crossover frequency). Substitute this frequency on real part and find the crossing point of the locus on real axis. Sketch the approximate locus of G(jω)H(jω) from the knowledge of type number and order of the system (or from the value of G(jω)H(jω) at ω = 0 and ω = infinite).

   (or)

   (iii) Separate the magnitude and phase of G(jω)H(jω). Equate the phase of G(jω)H(jω) to -180º and solve for ω. This value of ω is the phase crossover frequency and the magnitude at this frequency is the crossing point on real axis. Sketch the approximate root locus as mentioned in method (ii).

4. The section C_2 of Nyquist contour has a semicircle of infinite radius. Therefore, every point on section C_2 has infinite magnitude but the argument varies from $+\pi/2$ to $-\pi/2$. Consider the loop transfer function in time constant form and with y number of poles at origin, as shown below. Let G(s)H(s) has m zeros & n poles including poles at origin. For practical systems, n>m. From the above two equations we can conclude that the section C_2 of Nyquist contour in s-plane is mapped as circles/circular are around origin with radius tending to zero in the G(s)H(s)-plane.

5. In section C3, the value of ω varies from -∞ to 0. The mapping of section C3 is obtained by letting s=jω in G(s)H(s) and varying ω from -∞ to 0. The locus of G(jω)H(jω) as ω is varied from -∞ to 0 will be the G(s)H(s)-contour in G(s)H(s)-plane corresponding to section C3 in s-plane. This locus is the inverse polar plot of G(jω)H(jω). The inverse polar plot is given by the mirror image of polar plot with respect to real axis.
6. The section C4 of Nyquist contour has a semicircle of zero radius. Therefore, every point on semicircle has zero magnitude but the argument varies from -π/2 to π/2. Hence the mapping of section C4 from s-plane to G(s)H(s)-plane can be obtained by letting in G(s)H(s) and varying θ from -π/2 to π/2.

PERFORMANCE CRITERIA

For ordinary random inputs (i.e. inputs such that the error E is a stationary random function of time t), it is usual to adopt the mean -square- error as the performance criterion. This is the analogue of integral- square-error for simple transient inputs.